PHYS 105: Introduction to Computational Physics

Spring 2016

Homework #5

(Due: May 19, 2016)

1. Consider the 1-dimensional motion of a particle moving under a harmonic force:

$$a(x) = -k\,x\,,$$

with $k = 1$ and $x = 0$, $v = 1$ at $t = 0$. Now friction also acts on the particle, producing an acceleration proportional to, and always opposite, the velocity:

$$a_f = -\alpha v\,.$$

For small values of α, the mathematical solution to the equations of motion may be shown to be of the form

$$x = A\,e^{-bt}\,\sin\omega' t\,.$$

(a) For $\alpha = 0.1$, and taking time steps $\delta t = 0.001$, determine the period T by finding the first time (after $t = 0$) at which the particle crosses $x = 0$ with $v > 0$ (using linear interpolation, as usual, to refine the answer). Hence determine the value of ω'.

(b) By considering the decrease in amplitude from one maximum (or minimum) to the next, determine the ratio r by which the amplitude decreases from one peak to the next. Hence calculate the value of b for this α.

(c) As α increases, the oscillations are more strongly damped, and eventually cease. Determine, to two significant figures, the critical value of α beyond which there are no oscillations (i.e. the particle never crosses $x = 0$ again).

As usual, turn in your program, any non-graphical output produced when it runs, and all requested plots and additional calculations.

2. [In-class Exercise 7.2] Consider the motion of the nonlinear, damped, driven pendulum, described by the equation

$$a \;\equiv\; \frac{d^2x}{dt^2} \;=\; -\,k\sin x - \alpha v + g\cos(\omega_D t)\,.$$

Solve the above equations numerically and plot the phase-space (x–v) trajectories for $x_0 = v_0 = 1$, with $k = 1$, $\omega_D = 2/3$, $dt - 0.001$ and the following choices of α and g:

$$\begin{aligned}
\alpha &= 0.0, \quad g = 0\\
\alpha &= 0.5, \quad g = 0\\
\alpha &= 0.5, \quad g = 1\\
\alpha &= 0.5, \quad g = 1.07\\
\alpha &= 0.5, \quad g = 1.15\\
\alpha &= 0.5, \quad g = 1.5
\end{aligned}$$

Start each calculation at time $t = 0$, begin plotting trajectories at $t = 250$ to allow transients to die away, and continue each calculation to $t = 1500$. In all cases, "wrap" the variable x so that it always lies in the range $[-\pi, \pi]$, as follows:

```
while (x >  M_PI) x -= 2*M_PI;
while (x < -M_PI) x += 2*M_PI;
```

or

```
while x >  math.pi) x -= 2*math.pi
while x < -math.pi) x += 2*math.pi
```

Draw each trajectory as a separate graph, and choose axes with $-3.5 \leq x \leq 3.5$, $-3 \leq y \leq 3$.

3. Modify the program in problem 2 to produce *Poincaré sections* of the motion, that is, only print out the state of the system when the driving phase $\phi = \omega_D t$ is an integral multiple of 2π. Operationally, this is accomplished by checking at the end of each time step to see if the old and new values of ϕ straddle a multiple of 2π, and then using *linear interpolation* to determine the values of x and v at the proper phase.

   (a) Repeat the calculations in the previous question, plotting the results with the same axes as before. In the chaotic cases, continue the motion until $t = 10000$ in order to obtain a reasonable number of output points. (Note: this may take some time if you are programming in Python.)

   In the case $g = 1.15$, repeat the calculation for Poincaré sections at phases $2n\pi + \pi/4$ and $2n\pi + \pi/2$ (for integral n).

   (b) For the chaotic attractor with $g = 1.15$, choose an interesting region of the Poincaré section and "zoom in" on it to show its (fractal) structure in more detail . You will need to increase the length of your calculations as you proceed in order to maintain the number of plotted points. Turn in the plots obtained by zooming in on your chosen region by factors of 10 and 100.

   Note: usse the wwork done in class as much as possible, or at least modify the software the least possible.