PHYS 105: Introduction to Computational Physics

Spring 2016

Homework #3

(Due: April 28, 2016)

Write a program to perform a numerical integration of the trajectory of a particle moving in *one* dimension. The particle's position and velocity are updated at each time step according to the rules discussed in class:

$$
\begin{aligned}
t_{i+1} &= t_i + \delta t \\
x_{i+1} &= x_i + v_i\,\delta t + \tfrac{1}{2}\,a_i\,\delta t^2 \\
v_{i+1} &= v_i + a_i\,\delta t
\end{aligned}
$$

where the acceleration $a_i = a(x_i, v_i, t_i)$ is *recomputed at the start of each step*, allowing the possibility of arbitrarily complex motion.

1. First prove *analytically* that the above rule is *exact* for the case of constant acceleration. That is, if $t_i = i\,\delta t$ and

   $$
   \begin{aligned}
   x_i &= x_0 + v_0\,t_i + \tfrac{1}{2}\,a\,t_i^2 \\
   v_i &= v_0 + a\,t_i
   \end{aligned}
   $$

   for fixed a, show that the rule applied to (x_i, v_i, t_i) gives

   $$
   \begin{aligned}
   x_{i+1} &= x_0 + v_0\,t_{i+1} + \tfrac{1}{2}\,a\,t_{i+1}^2 \\
   v_{i+1} &= v_0 + a\,t_{i+1}
   \end{aligned}
   $$

   Note: this is an *algebraic* calculation, not a numerical one. Demonstrating that the result is true for a program with some choice(s) of parameters does *not* constitute a proof!

   Now use your program to answer the following questions.

2. Consider the force law

   $$a(x) = -k\,x\,,$$

   with $k = 4$, corresponding to *simple harmonic motion* with natural frequency $\omega_0 = 2$.

   (a) With initial conditions $x_0 = 0$, $v_0 = 1$ at time $t = 0$, follow the particle's motion with time step $\delta t = 0.05$. Plot the particle's trajectory (x versus t) for $0 \le t \le 4\pi$.

   (b) What is the exact (analytical) solution to this motion?

   (c) Is the "amplitude" of the numerically determined motion (i.e. the absolute value of x at each successive maximum or minimum) constant, as expected? If not, by what factor does the amplitude change from peak to peak for $\delta t = 0.05$. Try decreasing δt until the amplitude remains constant to within 1 percent from one cycle to the next. (You can estimate the amplitude “by eye” from a graph or modify your program to determine it more accurately by looking for instants when $v = 0$.) What is the value of δt thus obtained?

(d) Write down an expression for the instantaneous *total* energy (kinetic plus potential) E of the particle. This should be a conserved quantity of the motion. Plot the energy error $E(t) - E(0)$ as a function of time for $0 \le t \le 4\pi$ for the time step δt obtained in part (c).

(e) Using the value of δt determined in part (c), compute the *period* of the motion by finding the time taken for the particle to reach $x = 0$ again with $v > 0$. Use linear interpolation to refine your estimate, and compare your result with the analytic solution.

(f) Repeat part (e) for $v_0 = 0.25, 0.5, 1, 2, 4, 8$, and plot the numerically obtained period as a function of $E(0)$.

3. Repeat parts (c)–(f) of question 2 for the nonlinear force law

$$a(x) = -k\, x^3\,,$$

again with $k = 4$.

In addition to answers to specific questions, for problems 2 and 3, turn in hardcopy of your programs and (NON-graphical) output produced when they run, as well as all requested plots, *with axes clearly labeled.*