PHYS 501: Mathematical Physics I

Fall 2022, Homework #6

(Due December 6, 2022)

1. Poisson's equation (in <u>three</u> dimensions) is

$$\nabla^2 \phi = 4\pi G \rho.$$

(a) Let $\tilde{\phi}(\mathbf{k})$ and $\tilde{\rho}(\mathbf{k})$ be the Fourier transforms of $\phi(\mathbf{x})$ and $\rho(\mathbf{x})$, respectively. Show that

$$\tilde{\phi} = -\frac{4\pi G \tilde{\rho}}{k^2},$$

and hence write down an integral expression for $\phi(\mathbf{x})$.

(b) For a point mass at the origin, $\rho(\mathbf{x}) = m\delta(\mathbf{x})$. Use the result of part (a) to determine the solution for ϕ.

2. Find the Green's function $G(x, x')$ for the equation

$$\frac{d^2 y}{dx^2} - k^2 y = f(x)\,,$$

for $0 \leq x \leq L$, with $y(0) = y(L) = 0$. (Find G by solving the differential equation, not just as a formal sum over eigenfunctions!)

3. By first considering the behavior of the fundamental solution in the vicinity of $\mathbf{x} = \mathbf{x}'$, show that the Green's function $G(\mathbf{x}, \mathbf{x}')$ for the three-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = 0\,,$$

with the boundary condition that $u(\mathbf{x})e^{-i\omega t}$ represents outgoing waves at infinity, is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{e^{ikr}}{4\pi r}\,,$$

where $r = |\mathbf{x} - \mathbf{x}'|$.

4. In using the method of images to find the Dirichlet Green's function for Poisson's equation inside a sphere of radius a, it can be shown that the image of a point $\mathbf{x}'$ within the sphere is $\mathbf{x}'_1 = \alpha \mathbf{x}'$, with strength $-\beta$, so that the Green's function is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{\beta}{4\pi|\mathbf{x} - \mathbf{x}'_1|}.$$

(a) The boundary condition on G is that $G(\mathbf{x}, \mathbf{x}') = 0$ if $\mathbf{x}'$ lies on the surface of the sphere. Since $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$, it follows that $G(\mathbf{x}, \mathbf{x}')$ is also 0 if $\mathbf{x}$ lies on the surface of the sphere. By applying this latter condition at the two points $\mathbf{x}$ where the diameter through $\mathbf{x}'$ intersects the surface of the sphere, show that $\beta = a/r'$ and $\alpha = \beta^2$, where $r' = |\mathbf{x}'|$.

(b) Hence derive an expression for the solution $u(r, \theta, \phi)$ to Laplace's equation $\nabla^2 u = 0$ inside the sphere, subject to the boundary condition $u(a, \theta, \phi) = f(\theta, \phi)$.

(c) Compare this form of the solution with the series solution obtained by separation of variables within the sphere $r < a$.