PHYS 501: Mathematical Physics I

Fall 2022

Solutions to Homework #6

1. (a) Fourier transforming the equation gives

$$-k^2\tilde{\phi}(k) = 4\pi G\tilde{\rho}(k),$$

so

$$\tilde{\phi} = -\frac{4\pi G\tilde{\rho}}{k^2}$$

and the solution is

$$\phi(\mathbf{x}) = -4\pi G(2\pi)^{-3/2}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\tilde{\rho}(k)}{k^2}.$$

(b) If $\rho(\mathbf{x}) = m\delta(\mathbf{x})$, $\tilde{\rho} = (2\pi)^{-3/2}m$, so

$$\begin{aligned}
\phi &= -\frac{4\pi Gm}{(2\pi)^3}\int d^3k\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2} \\
&= -\frac{4\pi Gm}{(2\pi)^3}\int k^2dk\sin\theta_k d\theta_k d\phi_k\frac{e^{ikr\cos\theta_k}}{k^2},
\end{aligned}$$

where we have taken the "z axis" in k space to run parallel to $\mathbf{x}$, as usual. Doing the ϕ_k integral, setting $\mu = \cos\theta_k$, and simplifying, we find

$$\begin{aligned}
\phi &= -\frac{Gm}{\pi}\int_0^\infty dk\int_{-1}^1 e^{ikr\mu}d\mu \\
&= -\frac{2Gm}{\pi}\int_0^\infty dk\,\frac{\sin kr}{kr} \\
&= -\frac{Gm}{\pi}\int_{-\infty}^\infty dk\,\frac{\sin kr}{kr} \\
&= -\frac{Gm}{\pi r}\int_{-\infty}^\infty dz\,\frac{\sin z}{z} \\
&= -\frac{Gm}{r},
\end{aligned}$$

since the final integral has been shown in class to be π.

2. The Green's function $G(x,x')$ for the inhomogeneous ODE $y'' - k^2y = f(x)$ is determined by solving the differential equation with $f(x) = \delta(x-x')$ in $0 \le (x,x') \le L$, and matching solutions at $x = x'$ so that G is continuous and $[G']_-^+ = 1$. The boundary conditions are $y(0) = y(L) = 0$. In $0 \le x < x'$, the solution satisfying the boundary condition at $x = 0$ is

$$y(x) = C\sinh kx\,.$$

The corresponding solution in $x' < x \le L$ is

$$y(x) = C'\sinh k(x-L)\,.$$

The continuity and jump conditions at $x = x'$ are

$$\begin{aligned} C \sinh kx' &= C' \sinh k(x' - L) \\ Ck \cosh kx' &= C' k \cosh k(x' - L) - 1\,, \end{aligned}$$

so

$$\begin{aligned} C &= \frac{\sinh k(x' - L)}{k \sinh kL} \\ C' &= \frac{\sinh kx'}{k \sinh kL}\,, \end{aligned}$$

where we have used the identity

$$\sinh a \;\cosh b - \cosh a \;\sinh b = \sinh(a - b)\,.$$

Thus the Green's function is

$$\begin{aligned} G(x, x') &= \frac{\sinh kx \;\sinh k(x' - L)}{k \sinh kL},\; x < x' \\ &= \frac{\sinh k(x - L)\;\sinh kx'}{k \sinh kL},\; x > x'\,. \end{aligned}$$

3. Assume that the solution is a function of $\mathbf{x} - \mathbf{x}'$ and take $\mathbf{x}' = 0$ for convenience. Then the Green's function satisfies
$$\nabla^2 G + k^2 G = \delta(\mathbf{x}).$$
For $\mathbf{x} \neq 0$, we have $\nabla^2 G + k^2 G = 0$ and G is a sum of terms of the form
$$[a_l j_l(kr) + b_l n_l(kr)]\, Y_l^m(\theta, \phi).$$
Since $j_0(x) = \sin x / x$ and $n_0(x) = -\cos x / x$, we obtain the solution representing an outgoing spherical wave at infinity ($G \sim e^{ikr}/r$) by adopting spherical symmetry ($l = m = 0$) and choosing $b_0 = i a_0$ (so $G = -i b_0 h_0^{(1)}(kr)$, where $h_0^{(1)} = j_0 + i n_0$ is a Hankel function). Near $r = 0$,
$$G \;\sim\; b_0\, n_0(kr) \;\sim\; -\frac{b_0}{kr}.$$
Integrating the differential equation over an infinitesimal sphere centered on the origin, assuming G is continuous, and applying the divergence theorem to the $\nabla^2 G$ term as discussed in class, we find, near $r = 0$,
$$\begin{aligned} \frac{\partial G}{\partial r} &\sim \frac{1}{4\pi r^2} \\ \Rightarrow \qquad G &\sim -\frac{1}{4\pi r}. \end{aligned}$$
The two expressions for $G(r \to 0)$ are consistent if
$$b_0 = \frac{k}{4\pi}.$$
so
$$G = -\frac{e^{ikr}}{4\pi r} = -\frac{ik h_0^{(1)}(kr)}{4\pi}.$$

4. The Green's function is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} + \frac{\beta}{4\pi|\mathbf{x}-\mathbf{x}'_1|},$$

where $\mathbf{x}'_1 = \alpha\mathbf{x}'$ is the image point.

(a) We apply the boundary condition $G(\mathbf{x}, \mathbf{x}') = 0$ when $r = |\mathbf{x}| = a$ at the two points $\mathbf{x}_A = a\mathbf{x}'/r'$ and $\mathbf{x}_B = -a\mathbf{x}'/r'$, where the diameter through $\mathbf{x}'$ intersects the surface of the sphere. When $\mathbf{x} = \mathbf{x}_A$, we have $|\mathbf{x}-\mathbf{x}'| = a - r'$, $|\mathbf{x}-\mathbf{x}'_1| = \alpha r' - a$, so setting $G = 0$ implies

$$\frac{-1}{a-r'} + \frac{\beta}{\alpha r' - a} = 0,$$

or

$$\beta(a - r') = \alpha r' - a.$$

Similarly, when $\mathbf{x} = \mathbf{x}_B$, we have

$$\beta(a + r') = \alpha r' + a.$$

The solutions to these two equations are easily seen to be

$$\beta = \frac{a}{r'}, \qquad \alpha = \frac{a^2}{(r')^2} = \beta^2\,.$$

We assume without proof that G is in fact zero whenever $r = a$. Note that both α and β are 1 when $r' = a$, so $G(\mathbf{x}, \mathbf{x}') = 0$ then also.

(b) The solution to $\nabla^2 u = 0$ with $u(a, \theta, \phi) = f(\theta, \phi)$ is then

$$u(r, \theta, \phi) = \int a^2 d\Omega'\, f(\theta', \phi') \left.\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial r'}\right|_{r'=a},$$

where $d\Omega' = \sin\theta' d\theta' d\phi'$. Writing $\rho = |\mathbf{x}-\mathbf{x}'|$, $\rho_1 = |\mathbf{x}-\mathbf{x}'_1|$, and noting that

$$\begin{aligned}
\rho^2 &= (r')^2 + r^2 - 2r'r\cos\gamma, \\
\text{where} \quad \cos\gamma &= \frac{\mathbf{x}'\cdot\mathbf{x}}{r'r} \\
&= \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi'-\phi),
\end{aligned}$$

it follows that

$$\frac{\partial\rho}{\partial r'} = \frac{r' - r\cos\gamma}{\rho}$$

Similarly,

$$\begin{aligned}
\rho_1^2 &= (\alpha r')^2 + r^2 - 2\alpha r' r\cos\gamma, \\
&= a^4/(r')^2 + r^2 - 2a^2 r\cos\gamma/r', \\
\text{so} \quad 2\rho_1\frac{\partial\rho_1}{\partial r'} &= -2a^4/(r')^3 + 2a^2 r\cos\gamma/(r')^2 \\
\Rightarrow \quad \frac{\partial\rho_1}{\partial r'} &= -\frac{1}{\rho_1}\frac{a^3}{(r')^2}\left[\frac{a}{r'} - \frac{r}{a}\cos\gamma\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right)_{r'=a} &= -\frac{a - r\cos\gamma}{\rho^3}, \\
\frac{\partial}{\partial r'}\left(\frac{1}{\rho_1}\right)_{r'=a} &= \frac{a - r\cos\gamma}{\rho_1^3}.
\end{aligned}$$

Substituting in, we have

$$\begin{aligned}
\left.\frac{\partial G}{\partial r'}\right|_{r'=a} &= -\frac{1}{4\pi}\frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right) + \frac{\beta}{4\pi}\frac{\partial}{\partial r'}\left(\frac{1}{\rho_1}\right) \\
&= \frac{1}{4\pi}\left(\frac{a - r\cos\gamma}{\rho^3}\right) + \frac{\beta}{4\pi}\left(\frac{a - r\cos\gamma}{\rho_1^3}\right) \\
&= \frac{1}{2\pi\rho^3}\left(a - r\cos\gamma\right),
\end{aligned}$$

where have used the facts that $\beta = 1, \rho_1 = \rho$ when $r' = a$. Hence

$$u(r,\theta,\phi) = \frac{1}{2\pi}\int d\Omega'\, f(\theta',\phi')\left(\frac{a}{\rho}\right)^3\left(1 - \frac{r}{a}\cos\gamma\right).$$

(c) The series solution to the problem is

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} r^l Y_l^m(\theta,\phi),$$

where

$$a_{lm}a^l = \int d\Omega'\, f(\theta',\phi')\, Y_l^{m*}(\theta',\phi'),$$

so

$$u(r,\theta,\phi) = \sum_{l,m}\left(\frac{r}{a}\right)^l \int d\Omega'\, f(\theta',\phi')\, Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi).$$

We can connect this to the Green's function solution as follows. Using the addition theorem for $r < a$, $r_1 > a$, $r' \approx a$, we can expand $1/\rho$ and $1/\rho_1$ as

$$\begin{aligned}
\frac{1}{\rho} &= \sum_{l,m}\frac{4\pi}{2l+1}Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\frac{r^l}{(r')^{l+1}}, \\
\frac{1}{\rho} &= \sum_{l,m}\frac{4\pi}{2l+1}Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\frac{r^l}{(r_1)^{l+1}}
\end{aligned}$$

(with the same θ' and ϕ'). The Green's function thus is

$$\begin{aligned}
G &= -\sum_{l,m}\frac{1}{2l+1}Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\left[\frac{r^l}{(r')^{l+1}} - \beta\frac{r^l}{r_1^{l+1}}\right] \\
&= -\sum_{l,m}\frac{1}{2l+1}Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\,\frac{r^l}{(r')^{l+1}}\left[1 - \beta\alpha^{-(l+1)}\right] \\
&= -\sum_{l,m}\frac{1}{2l+1}Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\,\frac{r^l}{(r')^{l+1}}\left[1 - \left(\frac{r'}{a}\right)^{2l+1}\right] \\
&= -\sum_{l,m}\frac{1}{2l+1}Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\, r^l\left[(r')^{-l-1} - a^{-2l-1}(r')^l\right].
\end{aligned}$$

Hence

$$
\begin{aligned}
\left.\frac{\partial G}{\partial r'}\right|_{r'=a} &= -\sum_{l,m} \frac{1}{2l+1} Y_l^{m*}(\theta', \phi')\, Y_l^m(\theta, \phi)\, r^l \left[-(l+1)a^{-l-2} - a^{-2l-1} l a^l\right] \\
&= \frac{1}{a^2} \sum_{l,m} \left(\frac{r}{a}\right)^l Y_l^{m*}(\theta', \phi')\, Y_l^m(\theta, \phi), \\
\text{so} \qquad u(r, \theta, \phi) &= \int a^2 d\Omega'\, f(\theta', \phi')\, \left.\frac{\partial G}{\partial r'}\right|_{r'=a} \\
&= \sum_{l,m} \left(\frac{r}{a}\right)^l \int d\Omega'\, f(\theta', \phi')\, Y_l^{m*}(\theta', \phi')\, Y_l^m(\theta, \phi),
\end{aligned}
$$

in agreement with the series solution.