PHYS 501: Mathematical Physics I

Fall 2022, Homework #5 (Due November 22, 2022)

- 1. (a) Fill in the blanks, where u(x, y) and v(x, y) are the real and imaginary parts, respectively, of the analytic function w(z):
 - (i) $u(x,y) = e^{2x} \cos 2y$, v(x,y) = ?, w(z) = ?
 - (ii) u(x,y) =?, $v(x,y) = y(3x^2 y^2 2)$, w(z) =?
 - (iii) u(x,y) = ?, v(x,y) = ?, $w(z) = \tan^{-1} z$.

(b) Find *all* Laurent or Taylor expansions of the function

$$f(z) = \frac{z}{z^2 + 1}$$

about the point z = 2i, i.e. expand the function as a series of the form

$$f(z) = \sum_{n = -\infty}^{\infty} c_n \, s^n$$

where s = z - 2i. Note that there are several different regions in which different expansions apply.

2. Evaluate the following integrals using the residue theorem:

(a)
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$$
 (b) $\int_{0}^{\infty} \frac{x \sin kx \, dx}{x^2+1}$
(c) $\int_{-\infty}^{\infty} \frac{e^{ikx} \, dx}{(x^2+a^2)(x^2+b^2)}$ (d) $\int_{-\infty}^{\infty} \frac{x^2 e^x \, dx}{1+e^{2x}}$

where a and b are nonzero and $a \neq b$. In each case, sketch the contour you choose and clearly quote all theorems used in the derivation of your results.

3. Use contour integration to find the inverse Fourier transform f(t) of the function

$$F(\omega) = \sqrt{\frac{2}{\pi} \frac{\sin \omega a}{\omega}}$$

(where a > 0), for all values of t. (Use the " $(2\pi)^{-1/2}$ " version of the transform.) Recall that F was obtained as the Fourier transform of a step function with discontinuities at |t| = a. What are the values of f(-a) and f(a)? (Determine these values from the integral—don't appeal to the general properties of Fourier transforms!)

4. Find the (3-D) Fourier transform of the wave function for a 2p electron in a hydrogen atom:

$$\psi(\mathbf{x}) = (32\pi a_0^5)^{-1/2} z e^{-r/2a_0},$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius, r is radius, and z is a rectangular coordinate.