PHYS 501: Mathematical Physics I

Fall 2022, Homework #4

(Due November 10, 2022)

1. (a) Use the Legendre polynomial generating function

$$(1 - 2zh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z)h^n,$$

together with the binomial theorem, to derive expressions for the following quantities: (i) $P_n(1)$, (ii) $P_n(0)$, (iii) $P'_n(1)$, (iv) $P'_n(0)$.

(b) Use Stirling's formula

$$n! = \sqrt{2\pi n} n^n e^{-n} \left[1 + \mathcal{O}(1/n)\right]$$

to simplify the expression for $P_n(0)$ for even n as $n \to \infty$.

- (c) Evaluate the integral $\int_{-1}^{1} P_n(x) dx$.
- (d) Use a recurrence relation to evaluate the integral $\int_0^1 P_n(x) dx$.
- 2. A conducting sphere of radius a is surrounded by charged spherical shell of radius b. The inner sphere is held at potential Φ_0 . The outer shell has surface charge density

$$\sigma(\theta, \phi) = \sigma_0 \sin 2\theta \cos \phi$$

(in spherical coordinates r, θ, ϕ). The potential Φ satisfies Laplace's equation:

$$\nabla^2 \Phi = 0$$

for a < r < b and r > b, and is constrained by the requirements that

- (i) $\Phi = \Phi_0$ at r = a,
- (ii) $\Phi \to 0 \text{ as } r \to \infty$,
- (iii) Φ is continuous at r = b, and
- (iv) from Gauss's Law, the radial electric field $E_r = -\partial \Phi / \partial r$ has a discontinuity at r = b:

$$\left[-\frac{\partial\Phi}{\partial r}\right]_{b-}^{b+} = \frac{\sigma}{\epsilon_0}.$$

(a) Write down a general expression for Φ for a < r < b that satisfies Laplace's equation and condition (i).

(b) Write down a general expression for Φ for r > b that satisfies Laplace's equation and condition (ii).

(c) These expressions should take the form of sums over spherical harmonics $Y_l^m(\theta, \phi)$, for $-l \leq m \leq l$ and $l \geq 0$. The surface density σ can similarly be expressed as:

$$\sigma(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} Y_l^m(\theta,\phi).$$

Use the given expression for $\sigma(\theta, \phi)$ to determine the σ_{lm} .

(d) Conditions (iii) and (iv) must apply for <u>each</u> term in the sum over l and m. Apply them term by term at r = b to obtain the complete solution to the problem.

3. (a) Two hemispherical shells each of radius *a* are fitted together, insulated around their circle of contact, and kept at potentials $\pm V_0$, respectively. Find the potential $\Phi(r, \theta)$ inside the resulting sphere, where $\nabla^2 \Phi = 0$ inside the sphere, $\Phi = \pm V_0$ on the two hemispheres, and the polar axis $\theta = 0$ is the axis of symmetry.

(b) Now suppose that the potentials of the hemispheres in part (a) oscillate in time, with $V(t) = \pm V_0 e^{-i\omega t}$. Find the *exterior* solution $\Phi(r, \theta, t)$ to the wave equation

$$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

for r > a, with $\Phi = \pm V(t)$ on the hemispheres, that describes an *outgoing* wave as $r \to \infty$. [Recall that the asymptotic forms of the spherical Bessel functions $j_l(x)$ and $n_l(x)$ as $x \to \infty$ are $j_l(x) \sim \frac{1}{x} \cos\{x - \frac{\pi}{2}(l+1)\}$ and $n_l(x) \sim \frac{1}{x} \sin\{x - \frac{\pi}{2}(l+1)\}$.]

4. Each of the two 1S electrons in a helium atom may be described by a hydrogenic wave function

$$\psi(\mathbf{r}) = \left(\frac{8}{\pi a_0^3}\right)^{1/2} e^{-2r/a_0^2}$$

in the absence of the other electron. Here, $a_0 = \hbar^2/me^2$ is the Bohr radius. Use the expansion

$$\frac{1}{r_{12}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{2n+1} \left[Y_n^m(\theta_1, \phi_1) \right]^* Y_n^m(\theta_2, \phi_2) \begin{cases} \frac{r_1^n}{r_2^{n+1}}, & |\mathbf{r}_1| < |\mathbf{r}_2| \\ \frac{r_2^n}{r_1^{n+1}}, & |\mathbf{r}_1| > |\mathbf{r}_2| \end{cases}$$

to find the classical mutual electrostatic potential energy of the two electrons, treating each as a charge density distribution $\rho = -e|\psi|^2$:

$$U = \int \psi^*(\mathbf{r}_1) \psi^*(\mathbf{r}_2) \frac{e^2}{r_{12}} \psi(\mathbf{r}_1) \psi(\mathbf{r}_2) d^3 r_1 d^3 r_2,$$

where $r_i = |\mathbf{r}_i|, d^3r_i = r_i^2 dr_i \sin \theta_i d\theta_i d\phi_i$ and $r_{12} = |\mathbf{r}_1 - \mathbf{r}_1|.$