# PHYS 501: Mathematical Physics I

*Fall 2022, Homework #3*

(Due October 21, 2022)

1. A function $f(x)$ is periodic with period $2\pi$, and can be written as a polynomial $P(x)$ for $-\pi < x < a$ and as a polynomial $Q(x)$ for $a < x < \pi$. Show that the Fourier coefficients $c_n$ of $f$ go to zero at least as fast as $1/n^2$ as $n \to \infty$ if $P(a) = Q(a)$ (i.e. $f$ is continuous), but only as $1/n$ otherwise. (Hint: use integration by parts on the expression for $c_n$.)

2. (a) Find the Fourier series $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$, for $-1 < x < 1$, for the sawtooth function

$$f(x) = \begin{cases} -1 - x & (-1 < x < 0) \\ 1 - x & (0 < x < 1)\,. \end{cases}$$

(b) Plot the partial sums $S_N(x) = \sum_{n=1}^{N} b_n \sin(n\pi x)$ of the series for $0 \le x \le 1$, in steps of $\delta x = 0.0005$, and $N = 1$, 5, 10, 20, 50, 100, and 500. What is the maximum overshoot of the Fourier series relative to the original function in the $N = 500$ case, and at what value of $x$ does it occur?

3. (a) Find the series solution of the equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

that is regular at $x = 0$. Under what circumstances (for what values of $n$) does the series converge for *all* $x$?

(b) Find two linearly independent solutions of the equation

$$4x^2 y'' + (1 - p^2)y = 0\,.$$

(c) Given that one solution of the differential equation

$$y'' - 2xy' = 0$$

is $y(x) = 1$, use the Wronskian development to find a second, linearly independent solution. Describe its behavior near $x = 0$.

4. (a) Show explicitly from the series solutions that

$$\begin{aligned} J_{1/2}(x) &= A\,x^{-1/2} \sin x \\ J_{-1/2}(x) &= B\,x^{-1/2} \cos x\,. \end{aligned}$$

Hence, taking $A = B = 1$ and using the recurrence relation $J_{m+1}(x) = (2m/x)J_m(x) - J_{m-1}(x)$, write down expressions for $J_{3/2}(x)$ and $J_{5/2}(x)$.

(b) A function $f(x)$ is expressed as a Bessel series

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn} x),$$

where $\alpha_{mn}$ is the $n$-th root of $J_m$. Prove the Parseval relation

$$\int_0^1 [f(x)]^2 x\, dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 \left[J_{m+1}(\alpha_{mn})\right]^2 .$$

(You may assume the orthogonality relation $\int_0^1 J_m(\alpha_{mi}x) J_m(\alpha_{mj}x)\, x\, dx = \frac{1}{2} \left[J_{m+1}(\alpha_{mi})\right]^2 \delta_{ij}$ — see H&R §9.5.3.)