

# PHYS 501: Mathematical Physics I

## *Fall 2022, Homework #2*

(Due October 6, 2022)

1. (a) On performing separation of variables of Laplace's equation

$$\nabla^2 u = 0$$

in plane polar coordinates, with

$$u(r, \theta) = R(r)\Theta(\theta),$$

show that the radial function  $R(r)$  corresponding to angular dependence  $\Theta(\theta) = e^{im\theta}$  satisfies the ODE

$$r^2 R'' + rR' - m^2 R = 0,$$

and that this equation has solutions  $R = r^{\pm m}$ .

(b) Hence write down the general solution to Laplace's equation in polar coordinates.

(c) Find the solution  $u(r, \theta)$  of Laplace's equation inside a circle of radius  $a$ , where  $u$  is regular inside the circle and satisfies the boundary conditions

$$u(a, \theta) = U \cos^2 \theta.$$

2. (a) A particle of mass  $m$  is contained in a cylinder of radius  $R$  and height  $H$ . The particle is described by a wavefunction  $\psi(\rho, \phi, z)$  satisfying

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E\psi,$$

where  $\rho, \phi, z$  are cylindrical polar coordinates and  $\psi = 0$  on the surface of the cylinder ( $\rho = R, z = 0, H$ ). Find the ground-state energy of the system, and write down an explicit expression for the (unnormalized) lowest-energy wavefunction.

(b) Repeat part (a), but now for a particle moving in *two* dimensions, within a semicircular region of radius  $R$ , again with  $\psi = 0$  on the boundary of the region.

3. Find the three lowest-frequency modes of oscillation of acoustic waves in a hollow sphere of radius  $R$ . Assume a boundary condition  $\partial u / \partial r = 0$  at  $r = R$ , where  $u$  obeys the differential equation

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

4. The neutron density  $n$  inside a spherical sample of fissionable material obeys the equation

$$\nabla^2 n + \lambda n = \frac{1}{\kappa} \frac{\partial n}{\partial t},$$

where  $\lambda > 0$ ,  $\kappa > 0$ , and  $n = 0$  on the surface of the sample.

(a) Suppose the sample is spherical, of radius  $R$ . By seeking spherically symmetric modes with time dependence  $e^{\alpha t}$ , find the critical radius  $R_0$  such that  $n$  is unstable and *increases* exponentially with time for  $R > R_0$ .

(b) Now suppose the sample is a hemisphere, again of radius  $R$ . Repeat part (a), for axially symmetric modes.

(c) Two hemispheres of the material, each just barely stable as in part (b), are brought together to form a sphere. This sphere is unstable, with

$$n \sim e^{t/\tau}.$$

Find the time constant  $\tau$  of the resulting explosion.