PHYS 501: Mathematical Physics I

Fall 2022

Solutions to Homework #2

1. (a) Separating $u(r,\theta) = R(r)\Theta(\theta)$, Laplace's equation becomes

$$\frac{r}{R}\left(rR'\right)' + \frac{\Theta''}{\Theta} = 0.$$

Hence $\Theta''/\Theta = -m^2$, where m is an integer (by the usual argument) and

$$r\left(rR'\right)' - m^2 R = 0,$$

or

$$r^2R'' + rR' - m^2R = 0.$$

Seeking a power-law solution $R \sim r^\alpha$ and substituting in, we find $\alpha = \pm m$. For $m = 0$ the two solutions are the same, but going back to the underlying equations we find $(rR')' = 0$, so a second solution is $R = \log r$. In addition, $\Theta'' = 0$, so $\Theta = A\theta + B$, but requiring Θ to be single valued as θ goes from 0 to 2π implies $A = 0$.

(b) Hence the general solution is

$$u_m(r,\theta) = B\log r + \sum_m^\infty r^m\left(a_m\cos m\theta + b_m\sin m\theta\right).$$

(c) The boundary condition is $u(a,\theta) = U\cos^2\theta = \frac{1}{2}U(1+\cos 2\theta)$, which picks out the cosine terms with $m = 0$ and $m = 2$. Hence the interior regular solution (non-logarithmic and non-negative powers of r) is

$$u(r,\theta) = \tfrac{1}{2}U\left(1 + \frac{r^2}{a^2}\cos 2\theta\right).$$

2. (a) Schrödinger's equation is

$$(\nabla^2 + k^2)\psi = 0,$$

where $k^2 = 2mE/\hbar^2$. The boundary conditions are that $\psi = 0$ on all surfaces of a cylinder of radius R and height H. Take the axis of the cylinder to have $r = 0$ in cylindrical polar coordinates, and the flat faces to lie at $z = 0$ and $z = H$. The general form of the solution is a sum of terms of the form

$$\psi \sim J_m(\beta r)e^{im\phi}\sin lz,$$

where $\beta^2 + l^2 = k^2$ and the $\sin lz$ term is chosen to satisfy the boundary condition at $z = 0$. The boundary condition at $z = H$ then implies $lH = n\pi$, for integral n. The boundary condition at $r = R$ is $J_m(\beta R) = 0$, so $\beta R = \alpha_{mq}$, the q-th root of J_m. Hence

$$E_{mqn} = \frac{\hbar^2k_{mqn}^2}{2m} = \frac{\hbar^2}{2m}\left[\beta^2 + l^2\right] = \frac{\hbar^2}{2m}\left[\left(\frac{\alpha_{mq}}{R}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right]$$

for integral m, q, and n. Clearly the minimum energy corresponds to $m=0, q=1, n=1$, so

$$E_{min} = \frac{\hbar^2}{2m}\left[\left(\frac{\alpha_{01}}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right].$$

Here, $\alpha_{01} = 2.405$. The corresponding (unnormalized) wavefunction is

$$\psi \sim J_0\left(\frac{\alpha_{01}\, r}{R}\right) \sin\left(\frac{\pi z}{H}\right)$$

(b) In two dimensions, similar reasoning to that in the previous problem leads to the conclusion that the wavefunction must have the form

$$\psi \sim J_m(kr)\, e^{im\theta}.$$

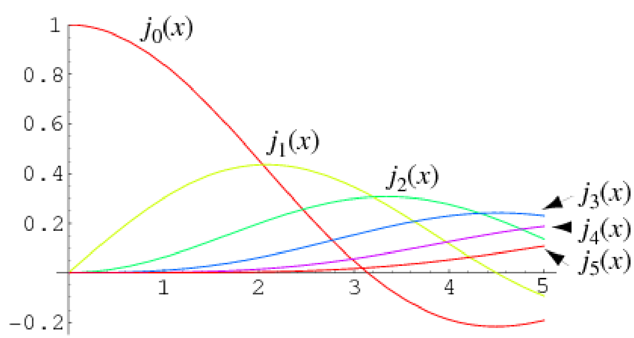
The boundary condition $\psi = 0$ at $r = R$ implies $J_m(kR) = 0$. The boundary condition at $\theta = 0, \pi$ implies that the appropriate $\sim e^{im\theta}$ term is actually $\sin m\theta$, where m is a positive integer. The minimum k, and hence E, occurs at the lowest nonzero root of J_m for $m > 0$, corresponding to the first root of J_1, $\alpha_{11} = 3.83$. Hence the ground-state solution (again unnormalized) has

$$\psi \sim J_1\left(\frac{\alpha_{11}\, r}{R}\right) \sin\theta, \qquad E = \frac{\hbar^2}{2m}\left(\frac{\alpha_{11}}{R}\right)^2.$$

3. The solutions to the wave equation in a sphere are of the form

$$u(r,\theta,\phi) = j_l(kr) P_l^m(\cos\theta) e^{im\phi},$$

for integer l and m. The boundary condition $\partial u/\partial r = 0$ at $r = R$ requires $j_l'(kR) = 0$. As illustrated in the figure below, the three lowest allowed values of kR correspond, respectively, to the first zeros of j_1' and j_2', and the second zero of j_0'.



Since

$$j_0(x) = \frac{\sin x}{x},$$

we have

$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2},$$

so $j_0'(x) = 0 \to \tan x = x$, or $x = 4.49$. Similarly, since

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \\ j_2(x) &= \sin x\left(\frac{3}{x^2} - 1\right) - \frac{3\cos x}{x^2}, \end{aligned}$$

$j_1'(x) = 0$ for $x = 2.08$, $j_2'(x) = 0$ for $x = 3.34$. (Note that the first zero of j_3' is at $x = 4.52$.) Thus, the three lowest frequencies are $\omega = kc = 2.08c/R, 3.34c/R, 4.49c/R$.

4. The equation to be solved is

$$\nabla^2 n + \lambda n = \frac{1}{\kappa}\frac{\partial n}{\partial t}\,,$$

where $\lambda, \kappa > 0$ and $n = 0$ on the surface. For assumed time dependence $n \sim e^{\alpha t}$, the equation becomes

$$\nabla^2 n + k^2 n = 0,$$

where $k^2 = \lambda - \alpha/\kappa$. The critical case has $\alpha = 0$, or $k^2 = \lambda$.

(a) For a sphere, the general solution is $n \sim j_l(kr)P_l^m(\cos\theta)\,e^{im\phi}$. The surface boundary condition is $j_l(kR) = 0$, and the minimum k corresponds to the first root of j_0, so $l = m = 0$. Since $j_0(x) \sim \sin x/x$, we find $kR = \pi$ and the critical radius is

$$R_0 = \frac{\pi}{k} = \frac{\pi}{\sqrt{\lambda}}\,.$$

Note that, in order to satisfy the boundary condition, increasing R has the effect of decreasing k and hence of increasing $\alpha = \kappa(\lambda - k^2)$. Thus the sphere is unstable for $R > R_0$.

(b) For a *hemisphere*, the extra boundary condition at $\theta = \pi/2$ means that the $l = 0$ mode is not a solution. We now require $P_l^m(\cos\theta) = 0$ at $\theta = \pi/2$ (where we have assumed that the z axis is the axis of symmetry of the hemisphere). The lowest-order P_l^m satisfying the boundary condition is $P_1^0 = \cos\theta$, so $l = 1$ and the radial boundary condition becomes $j_1(kR) = 0$. Since $j_1(x) \sim \sin x/x^2 - \cos x/x$, the first zero has $x = \tan x$, or $x = 1.43\pi = 4.49$. The critical ($\alpha = 0$) radius for this geometry then is

$$R_1 = \frac{1.43\pi}{k} = \frac{1.43\pi}{\sqrt{\lambda}} = 1.43R_0\,.$$

(c) Now the system is spherical again, but the radius is $R_1 > R_0$ and the system is unstable. Writing $\beta = 1.43$, the boundary condition now implies

$$\begin{aligned} kR_1 = \left(\lambda - \frac{\alpha}{\kappa}\right)^{1/2} R_1 \quad &= \quad \pi \\ \Rightarrow \quad \alpha \quad &= \quad \kappa\lambda\left(1 - \beta^{-2}\right). \end{aligned}$$

The growth time scale therefore is

$$\tau = \alpha^{-1} = \left(\frac{\beta^2}{\beta^2 - 1}\right)\frac{1}{\kappa\lambda} = \frac{1.96}{\kappa\lambda}\,.$$