PHYS 431/531: Galactic Astrophysics

Fall 2021

Solutions to Homework #6

1. (a) The Kuzmin potential is

$$\Phi_K(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}.$$

For $z > 0$ denote the denominator by $s = (R^2 + (a+z)^2)^{1/2}$. Then

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial R^2} \\
&= GM\left[\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R^2}{s^3}\right) + \frac{\partial}{\partial z}\left(\frac{a+z}{s^3}\right)\right] \\
&= GM\left[\frac{2}{s^3} - \frac{3R^2}{s^5} + \frac{1}{s^3} - \frac{3(a+z)^2}{s^5}\right] \\
&= \frac{GM}{s^5}\left[3s^2 - 3\{R^2+(a+z)^2\}\right] \\
&= 0,
\end{aligned}$$

and similarly for $z < 0$. Φ is not differentiable at $z = 0$; the potential represents a mass sheet in that plane.

(b) Let the surface density of the sheet be $\Sigma(R)$, and consider a "Gaussian pillbox"—a right cylinder of cross-sectional area δA and negligible extent in the z direction, with axis parallel to the z axis—straddling the sheet. Let $a_z(R,z) = -\partial\Phi\partial z$ be the z component of the accdeleration. Clearly $a_z(R,-z) = -a_z(R,z)$ and, in particular, the acceleration just below the disk is equal and opposite to the acceleration just above it, $a_z(R,0_-) = -a_z(R,0_+)$. Hence, by Gauss's law, $-2a_z(R,0_+)\delta A = 4\pi G\Sigma(R)\delta A$, so

$$\begin{aligned}
\Sigma(R) &= -\frac{1}{2\pi G}\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} \\
&= \left.\frac{aM}{2\pi}s^{-3/2}\right|_{z=0} \\
&= \frac{aM}{2\pi}(R^2+a^2)^{-3/2}.
\end{aligned}$$

(c) The circular orbit speed is v_c, where

$$\begin{aligned}
v_c^2 &= -R\left.\frac{\partial\Phi}{\partial R}\right|_{z=0} \\
&= \left.\frac{GMR^2}{s^3}\right|_{z=0} \\
&= \frac{GMR^2}{(R^2+a^2)^{3/2}}.
\end{aligned}$$

2. The number density is
$$n(z) = \int_0^{v_{esc}} f(z, v_z) dv_z,$$
where we can take the upper limit to be infinite so long as $v_{esc} \gg \sigma$ (which we assume). Then
$$n(z) = \frac{n_0}{\sqrt{2\pi\sigma^2}} e^{-\Phi/\sigma^2} \int_0^\infty e^{-v_z^2/2\sigma^2} dv_z = n_0 e^{-\Phi(z)/\sigma^2}.$$
Thus n_0 is the number density in the plane, at $z = 0$.
Now let $\phi = \Phi/\sigma^2$ and $\rho = mn$. Then Poisson's equation $\Phi'' = 4\pi G\rho$ becomes
$$\left(\frac{\sigma^2}{4\pi G m n_0}\right) \frac{d^2\phi}{dz^2} = e^{-\phi},$$
so, setting $y = z/z_0$, where $z_0^2 = \sigma^2/8\pi G m n_0$, we have
$$2\frac{d^2\phi}{dy^2} = e^{-\phi}.$$
We can immediately integrate this once to find
$$(\phi')^2 = -e^{-\phi} + \text{constant}.$$
Choosing $\phi = \phi' = 0$ at $z = 0$ sets the constant equal to 1, and integrating again gives
$$y = \int_0^\phi \frac{dp}{\sqrt{1 - e^{-p}}}.$$
We can evaluate the integral by setting $u = e^{-p/2}$, so
$$y = \int_1^u \frac{-2du}{u\sqrt{1-u^2}} = 2\,\mathrm{sech}^{-1} u.$$
Hence
$$e^{-\phi/2\sigma^2} = -\mathrm{sech}\left(\frac{z}{2z_0}\right),$$
and
$$n(z) = n_0\,\mathrm{sech}^2\left(\frac{z}{2z_0}\right) \sim 4n_0\, e^{-|z|/z_0} \quad \text{as } z \to \infty.$$

3. For a distribution function $f(E)$, we have
$$\langle v_x^2 \rangle = \frac{1}{n} \int dv_x dv_y dv_z\, v_x^2\, f[\phi + \tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)],$$
while
$$\langle v_y^2 \rangle = \frac{1}{n} \int dv_x dv_y dv_z\, v_y^2\, f[\phi + \tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)].$$
Obviously these are the same, so it follows that $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$, and the velocity distribution is isotropic.

4. (a) By definition, $\mathcal{E} = \psi - \frac{1}{2}v^2$, $v^2 = v_r^2 + v_t^2$, and $L = rv_t$. Writing $X = L^2$, we can express the volume element in velocity space, $d^3v = 2\pi v_t dv_t dv_r$, as

$$2\pi v_t dv_t dv_r = 2\pi v_t \begin{vmatrix} \frac{\partial \mathcal{E}}{\partial v_t} & \frac{\partial \mathcal{E}}{\partial v_r} \\ \frac{\partial X}{\partial v_t} & \frac{\partial X}{\partial v_r} \end{vmatrix}^{-1} d\mathcal{E}dX = 2\pi v_t \begin{vmatrix} -v_t & -v_r \\ 2r^2 v_t & 0 \end{vmatrix}^{-1} d\mathcal{E}dX = \frac{\pi d\mathcal{E}dX}{r^2 v_r}.$$

(b) For a distribution function

$$f(\mathcal{E}, L) = \begin{cases} A\delta(L^2)(\mathcal{E} - \mathcal{E}_0)^{-1/2} & (\mathcal{E} > \mathcal{E}_0) \\ 0 & (\mathcal{E} \le \mathcal{E}_0), \end{cases},$$

we have

$$\rho = \int f\, d^3v = A \int\int d\mathcal{E}dX \left(\frac{\pi}{r^2 v_r}\right) \delta(X)(\mathcal{E} - \mathcal{E}_0)^{-1/2}$$

Now write $v_r^2 = 2(\psi - \mathcal{E}) - X/r^2$, and do the X integral to set $v_r^2 = 2(\psi - \mathcal{E})$, so

$$\rho(r) = \left(\frac{\pi A}{2\sqrt{2}}\right) r^{-2} \int_{\mathcal{E}_0}^{\psi(r)} (\mathcal{E} - \mathcal{E}_0)^{-1/2}(\psi - \mathcal{E})^{-1/2}\, d\mathcal{E}\,,$$

for $\psi(r) \ge \mathcal{E}_0$. By substituting $y = (\mathcal{E} - \mathcal{E}_0)/(\psi - \mathcal{E}_0)$, it is easily shown that the integral is simply $\int_0^1 y^{-1/2}(1-y)^{-1/2}\, dy = \pi$, which is independent of ψ and hence of r, so

$$\rho(r) = Br^{-2}\,,$$

for $r < r_0$, where $\psi(r_0) = \mathcal{E}_0$ and $B = \pi^2 A/2\sqrt{2}$.

5. (a) For a flat rotation curve, the Oort constants A and B satisfy

$$A + B = \frac{dV}{dR} = 0, \qquad A - B = \frac{V}{R} = \Omega,$$

where V is the rotation speed and Ω is the circular frequency. Hence $A = -B = \frac{1}{2}\Omega$. For the Galaxy, we have $V_0 = 220$ km/s, $R_0 = 8$ kpc, so $\Omega_0 = 27.5$ km/s/kpc. The epicyclic frequency κ is given by $\kappa^2 = -4B\Omega_0$, so $\kappa = \sqrt{2}\Omega = 39$ km/s/kpc $= 1/(25$ Myr$)$.

(b) For a star in epicyclic motion with radial amplitude X, the x and y velocities relative to the guiding center are (S&G, p. 138)

$$\begin{aligned} v_x &= -\kappa X \sin\psi, \\ v_y &= -\Omega X \cos\psi, \end{aligned}$$

where $\psi = \kappa t + \psi_0$ is the epicyclic phase. Solving for the observed velocities $v_x = -10$ km/s, $v_y = 5$ km/s, we find $X = 0.31$ kpc, $\psi = 125°$. The radius of the guiding center then is $R_g = R_0 - X\cos\psi = 8.2$ kpc.

(c) The figure `fig6.5c.png` shows Ω and the inner and outer Lindblad resonances for 2-armed (red) and 4-armed (green) spirals. The black line corresponds to a pattern speed of 30 km/s/kpc. It intersects the red curves at $R = 2.1$ and $R = 12.5$ kpc, and the green curves at $R = 4.7$ and $R = 10$ kpc. Thus the 2-armed spiral can propogate in an annulus of area 480 kpc^2, versus 240 kpc^2 for the 4-armed spiral. The exact numbers will of course depend on the pattern speed chosen, but the factor of roughly 2 persists.