PHYS 431/531: Galactic Astrophysics

Fall 2021

Solutions to Homework #5

1. The relaxation time is
$$t_r = \frac{v^3}{8\pi G^2 \langle m\rangle \rho \ln\Lambda},$$
where $v^2 = 3\sigma^2$. For the numbers in S&G Table 3.1, and $\langle m\rangle = 0.5\,M_\odot$, we find $t_r = 32\,\mathrm{Myr}$, $t_{cr} = 2r_c/\sigma = 0.089\,\mathrm{Myr} = 2.8\times10^{-3}\,t_r$. For $\langle m\rangle = 0.3\,M_\odot$, as in S&G, we find $t_r = 53\,\mathrm{Myr}$, $t_{cr} = 1.7\times10^{-3}\,t_r$. Note that we can't use the result $t_R \sim (N/\ln\Lambda)t_{cr}$ here, since that refers only to global averages.

2. The mean square velocity is
$$\begin{aligned}\langle v^2\rangle &= \frac{\int_0^\infty v^2 f(v)\,dv}{\int_0^\infty f(v)\,dv}\\ &= \frac{\int_0^\infty v^4 e^{-v^2/a^2}\,dv}{\int_0^\infty v^2 e^{-v^2/a^2}\,dv}\\ &= \frac{\frac{3}{8}\,\pi^{1/2}a^5}{\frac{1}{4}\,\pi^{1/2}a^3}\\ &= \tfrac{3}{2}a^2,\end{aligned}$$
where $a^2 = 2kT/m$. Hence $\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$, as claimed.

3. The cross section for a comet to enter the inner solar system is $\sigma = \pi b^2$, where b is given by the expression discussed in class,
$$b^2 = r_p^2\left(1 + \frac{2GM}{r_p V^2}\right).$$
with $r_p = 5.2$ AU and $V = 20$ km/s. Hence the "collision" rate is
$$n\sigma V = \pi r_p^2 n V\left(1 + \frac{2GM_\odot}{r_p V^2}\right) = 1\mathrm{yr}^{-1},$$
so
$$n = \frac{1\mathrm{yr}^{-1}}{\pi r_p^2 V\left(1 + \frac{2GM_\odot}{r_p V^2}\right)} = 1.3\times10^{13}\mathrm{pc}^{-3}.$$

4. (a) In virial equilibrium, $E = \frac{1}{2}U \propto M^2/R$. Since energy is conserved by evaporation, $E =$ constant, so $R \propto M^2$.
(b) Since $R \propto M^2$ and $t_R \propto M^{1/2}R^{3/2}$, we can write
$$\frac{t_r}{t_{R0}} = \left(\frac{M}{M_0}\right)^{7/2}.$$

The mass of the cluster obeys the equation

$$\frac{dM}{dt} = -\frac{M}{\alpha t_R} \, .$$

Writing $m = M/M_0$ and $\tau = t/t_{R0}$, this becomes

$$\frac{dm}{d\tau} = -m^{-5/2}/\alpha,$$

the solution to which is

$$\tfrac{2}{7}(m^{7/2} - 1) = -\tau/\alpha,$$

or

$$m = \left(1 - \frac{7\tau}{2\alpha}\right)^{2/7} .$$

The lifetime of the cluster (when $m = 0$) is therefore $\tau = \frac{2}{7}\alpha$, or $t = \frac{2}{7}\alpha t_{R0}$. The mean density $\bar{\rho} \propto M/R^3 \propto M^{-5}$, so

$$\bar{\rho} = \bar{\rho}_0 \left(1 - \frac{7\tau}{2\alpha}\right)^{-10/7} .$$

(c) For $M_0 = 5\times 10^5\, M_\odot$, $R_0 = 10$ pc, and a mean stellar mass of $0.5\, M_\odot$, the initial relaxation time is

$$t_{R0} \;\sim\; \frac{N_0}{6\sqrt{2}\ln\Lambda}\left(\frac{GM_0}{R_0^3}\right)^{-1/2} \;=\; 13.4 \text{ Gyr},$$

where we have taken $\Lambda = \frac{1}{2}N_0$ and $N_0 = 10^6$. Hence, taking $\alpha = 136$, we find $t = 520$ Gyr.