PHYS 431/531: Galactic Astrophysics

Fall 2021

Solutions to Homework #4

1. (a) The Jeans length and mass are given by

$$\lambda_J^2 = \frac{\pi c_s^2}{G\rho}, \qquad M_J = \frac{4\pi}{3}\rho\left(\tfrac{1}{2}\lambda_J\right)^3,$$

where the density here is $\rho = 1.1$ kg/m^3 and the sound speed is $c_s = 345$ m/s. Hence $\lambda_J = 7.14 \times 10^7$ m $= 71,400$ km, $M_J = 6.23 \times 10^{23}$ kg $= 3.16 \times 10^{-7} M_\odot$.

(b) The dispersion relation is

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho.$$

For a sound wave with $\lambda = 10\,\text{m}$, $k = 0.2\pi\,\text{m}^{-1}$ and the two terms on the right-hand side of the above relation are $w_0^2 = 4.7 \times 10^4$ and 9.2×10^{-10}, respectively, where the zero-gravity angular frequency is $\omega_0 = c_s k = 2.2 \times 10^2\,\text{s}^{-1}$. The change in frequency is

$$\delta\omega \;=\; \sqrt{\omega_0^2 - 4\pi G\rho} - \omega_0 \;\approx\; -\frac{2\pi G\rho}{\omega_0} = -2.1 \times 10^{-12}\,\text{s}^{-1},$$

so $\delta f = \delta\omega/2\pi = -3.4 \times 10^{-13}$ Hz.

(c) If $\lambda = \frac{1}{4}\lambda_J$, then $k^2 = 16k_J^2 = 64\pi G\rho/c_s^2$. Substituting this value of k into the above dispersion relation, we find

$$\omega^2 \;=\; c_s^2 k^2 - 4\pi G\rho \;=\; 60\pi G\rho.$$

The "short-wavelength" result is

$$\omega_s^2 \;=\; c_s^2 k^2 \;=\; 64\pi G\rho.$$

Hence

$$\frac{\omega}{\omega_s} \;=\; \sqrt{\frac{15}{16}} = 0.968, \quad \text{so} \quad \frac{\omega - \omega_s}{\omega_s} \;=\; -\,0.032.$$

2. (a) Neglecting, as usual, the distinction betwen the half-mass radius and the virial radius, the virial theorem estimate of the mass is $M = 2R\langle v^2\rangle/G = 5.2 \times 10^5\,M_\odot$. If we interpret the given 10 km/s as the 1-D (line of sight) velocity dispersion, the answer increases by a factor of 3, to $1.6 \times 10^6\,M_\odot$.

(b) Writing $\frac{1}{2}M = \frac{4\pi}{3}\langle m\rangle nR^3 = R\langle v^2\rangle/G$, we find $R = \sqrt{\frac{3\langle v^2\rangle}{4\pi Gn\langle m\rangle}} = 0.54$ pc, so $M = 1010\,M_\odot$.

(c) For $t_D = 2\pi\left(\frac{GM}{R^3}\right)^{-1/2} = 10^6$ yr and $R = 10\,\text{pc}$, we have $M = \frac{4\pi^2 R^3}{Gt_D^2} = 8.8 \times 10^6\,M_\odot$. However, this answer depends sensitively on the definition of t_D—if we omit the 2π, the mass becomes $M = 2.2 \times 10^5\,M_\odot$

3. (a) The total thermal energy is

$$E_T = \int_0^R 4\pi r^2 \rho(r) \frac{kT}{m}\, dr.$$

Since the pressure $P = nkT = \rho kT/m$, it follows that

$$E_T = \int_0^R 6\pi r^2 P(r)\, dr.$$

(b) Integrating by

$$\int_0^R r^2 P(r)\, dr = \left[\tfrac{1}{3} r^3 P\right]_0^R - \int_0^R \tfrac{1}{3} r^3 P'\, dr,$$

where we have used the fact that $P(R) \approx 0$. The equation of hydrostatic equilibrium tells us that $P' = -G\rho M(r)/r^2$, so

$$\begin{aligned} E_T &= -2\pi \int_0^R r^3 (-\rho) \frac{GM(r)}{r^2}\, dr \\ &= -\tfrac{1}{2} \int_0^R 4\pi r^2 \rho \left(\frac{-GM(r)}{r} \right) dr \\ &= -\tfrac{1}{2} U, \end{aligned}$$

so the virial theorem holds: $2E_T + U = 0$.

4. In virial equilibrium, the (3-D) rms velocity of stars is

$$\langle v^2 \rangle = \frac{GM}{2R}\,,$$

so for $R = 100$ kpc and $M = 10^{12}\, M_\odot$, we have $\langle v^2 \rangle^{1/2} = 268$ km/s. If this speed is representative of the gas, with $\frac{3}{2} kT = \frac{1}{2} m \langle v^2 \rangle$ (where m is the mass of a proton), the corresponding temperature is

$$T = \frac{m\langle v^2 \rangle}{3k} = \frac{GMm}{6kR} = 2.9 \times 10^6 K.$$