PHYS 431/531: Galactic Astrophysics

Fall 2021

Solutions to Homework #3

1. (a) The average speed is

$$\bar{v} \;=\; \sqrt{\frac{3kT}{m_H}} \;=\; 1.6 \text{ km/s} \left(\frac{T}{100\,K}\right)^{1/2}.$$

(b) The typical atomic center-of-mass energy is

$$\tfrac{1}{2} m_H \bar{v}^2 \;=\; 2.1 \times 10^{-21}\,\text{J} \;=\; 0.013\,\text{eV}.$$

which is much greater than the energy of the $\lambda = 21$-cm line,

$$E_{21} = \frac{hc}{\lambda} \;=\; 9.5 \times 10^{-25}\,\text{J} \;=\; 5.9 \times 10^{-6}\,\text{eV}.$$

(c) At any instant, three-fourths of the hydrogen atoms are in the upper state. The total number of atoms is $N_H = M_H/m_H = 6.0 \times 10^{66}$. A mean deexcitation time of 1.1×10^7 yr means a deexcitation rate of $r = 1/1.1 \times 10^7\,\text{yr} = 2.88 \times 10^{-15}\,\text{s}^{-1}$ — that is, in one second, a fraction $f = 2.88 \times 10^{-15}$ of those atoms emit a photon. Thus the 21-cm luminosity is

$$L_{21} \;=\; \tfrac{3}{4} N_H\, r\, E_{21} \;=\; 1.2 \times 10^{28}\,\text{W} \;=\; 31\, L_\odot.$$

2. (a) The radiative energy flux at distance r from the star is $f = L/4\pi r^2$. The grain's projected area is $A_g = \pi r_d^2$, so it absorbs energy at a rate $\Gamma = f A_g = \frac{1}{4} L (r_d/r)^2$. It radiates at a rate of σT^4 per unit area, and its surface area is $4\pi r_d^2$, so the total blackbody luminosity is $\Lambda = 4\pi\sigma r_d^2 T^4$. Setting $\Gamma = \Lambda$ implies $T = (L/16\pi\sigma r^2)^{-1/4}$. For the given parameters, we find $T = 23.2$ K. The peak wavelength for a blackbody at this temperature is $\lambda_{max} = 2900\mu\text{m}/T = 125\mu\text{m}$.

(b) The fraction of area covered by dust (and hence the fraction of photons absorbed) between radii r and $r + dr$ is $(4\pi r^2 dr n_d) \times \pi r_d^2/4\pi r^2 = \pi r_d^2 n_d dr$. Assuming isotropic emission with N_0 photons injected at $r \approx 0$ and $N(r)$ representing the number of surviving photons at radius r, it follows that $N(r+dr) = N(r) - \pi r_d^2 n_d dr N(r)$. Taking the limit, we find $dN/dr = -\pi r_d^2 n_d N$, so $N(r) = N_0 \exp(-\pi r_d^2 n_d r) = N_0 \exp(-\tau)$. Putting in the numbers for $r = 2$ pc, we find $\tau = 0.65$ and $N(r) = 0.52 N_0$, so the fraction of photons escaping the cloud is 52%.

3. The sound speed is $c_s = \sqrt{\frac{\gamma k T}{m}}$, where m is the mean particle mass. The Jeans mass is $M_J = \frac{\pi}{6} \lambda_J^3\, n\, m_H$. In convenient units, with $\gamma = \frac{5}{3}$, we have

$$\begin{aligned} c_s &= 370\,\text{m/s} \left(\frac{T}{10\,K}\right)^{1/2} \left(\frac{m}{m_H}\right)^{-1/2} \\ M_J &= 3.5\, M_\odot \left(\frac{T}{10\,K}\right)^{3/2} \left(\frac{n}{10^6\,\text{cm}^{-3}}\right)^{-1/2} \left(\frac{m}{m_H}\right)^{-2}. \end{aligned}$$

(a) For molecular hydrogen, with $T = 10$ K, $n = 10^6\,\text{cm}^{-3}$, and $m = 2m_H$, $c_s = 260$ m/s, $M_J = 0.83\,M_\odot$.

(b) For atomic hydrogen, with $T = 100$ K, $n = 1\,\text{cm}^{-3}$, and $m = m_H$, $c_s = 1.2$ km/s, $M_J = 1.0 \times 10^5\,M_\odot$.

(c) For ionized hydrogen, with $T = 10^6$ K, $n = 10^{-3}\,\text{cm}^{-3}$, and $m = \frac{1}{2}m_H$, $c_s = 170$ km/s, $M_J = 1.3 \times 10^{13}\,M_\odot$.

4. For a particle moving within a homogeneous sphere of mass M and radius a, the mass inside radius r is $m(r) = \frac{4}{3}\pi M(r/a)^3$, so the equation of motion is

$$\ddot{r} + \frac{GM}{a^3}r = 0\,.$$

The solution satisfying the initial condition $r = a, \dot{r} = 0$ is

$$r = a\cos\Omega t\,,$$

where $\Omega^2 = GM/a^3$. Obviously $r = 0$ at time $t = t_1 = \pi/2\Omega = \frac{\pi}{2}\sqrt{\frac{a^3}{GM}} = \sqrt{\frac{3\pi}{16G\rho}}$.

5. If the entire sphere is collapsing, then for a point on the u;[=y surface, $m(r) = M$, so

$$\ddot{r} + \frac{GM}{r^2} = 0\,.$$

Writing the energy per unit mass as $E = \frac{1}{2}\dot{r}^2 - GM/r = -GM/a$, we have $\dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{a}\right)$, so the collapse time is

$$t_2 = \sqrt{\frac{a}{2GM}}\int_0^a \frac{dr}{\sqrt{a/r - 1}}\,.$$

Substituting $r = a\sin^2\theta$, we find

$$t_2 = 2a\sqrt{\frac{a}{2GM}}\int_0^{\pi/2}\sin^2\theta d\theta = \frac{\pi}{2}\sqrt{\frac{a^3}{2GM}} = t_1/\sqrt{2}\,.$$