PHYS 431/531: Galactic Astrophysics

Fall 2021

Solutions to Homework #2

1. (a) Assume that the sheet lies in the $x - y$ plane. Obviously by symmetry the acceleration is in the z direction and is independent of x and y. Write it as $a(z)$, where (again by symmetry) $a(-z) = -a(z)$. Now apply Gauss's law ($\int_V \nabla^2\phi = \int_S \nabla\phi \cdot d\mathbf{S}$) to a right cylinder with axis in the z direction, of cross-sectional area δA, and extending from $-z$ to $+z$. Only the ends of the cylinder contribute to the total flux, and the mass inside is $\Sigma\delta A$. Thus,

$$-2a(z)\delta A = 4\pi G\Sigma\delta A\,,$$

so $a(z) = -2\pi G\Sigma$ (for $z > 0$), independent of z.

(b) The maximum distance above the plane is $z_{max} = v_0^2/2|a|$, where v_0 is the z velocity as the star crosses the plane. Hence

$$|a| = v_0^2/2z_{max} = 2\pi G\Sigma\,,$$

so

$$\Sigma = v_0^2/4\pi G z_{max} = 0.070\,\mathrm{kg/m^2} = 33\,M_\odot/\mathrm{pc}^2\,.$$

2. (a) If the stellar density is

$$n(R,z) \;=\; n_0\, e^{-R/h_R}\, e^{-|z|/h_z},$$

then the surface luminosity is

$$\Sigma(R) \;=\; \int_{-\infty}^{\infty} L_* n(R,z)\, dz \;=\; 2L_* n_0 h_z e^{-R/h_R}.$$

(b) Hence, integrating over R,

$$L_G \;=\; 2L_* n_0\, h_z \int_0^\infty 2\pi R e^{-R/h_R} dR \;=\; 4\pi L_*\, n_0\, h_z\, h_R^2.$$

(c) Writing $2L_* n_0 h_z = L_G/2\pi h_R^2$, we have

$$\Sigma(R) \;=\; \frac{L_G}{2\pi h_R^2}\, e^{-R/h_R} \;=\; 27\ L_\odot\,\mathrm{pc}^{-2}\ \ \text{at } R = R_0$$

for $R_0 = 8$ kpc and $h_R = 4$ kpc.

(d) Since $n_0 = L_G/4\pi h_R^2\, h_z L_*$, if $L_* = L_\odot$ and $h_z = 250$ pc, the number density at $z = 0$ is

$$n(R_0, 0) \;=\; n_0\, e^{-R_0/h_R} \;=\; 0.054\,\mathrm{pc}^{-3}.$$

3. (a) From the definitions of A and B,

$$\begin{aligned} A &\equiv -\tfrac{1}{2}R\left(\frac{V}{R}\right)' &= -\tfrac{1}{2}V' + \tfrac{1}{2}\frac{V}{R} \\ B &\equiv -\tfrac{1}{2}\frac{(RV)'}{R} &= -\tfrac{1}{2}V' - \tfrac{1}{2}\frac{V}{R}. \end{aligned}$$

Hence $A + B = -V'$, $A - B = V/R$.

(b) If $R_0 = 8$ kpc and $A - B = 15.1 - (-13.4) = 28.5$ km/s/kpc, then $V_0 = 228$ km/s.

(c) For

$$\rho(R) = \rho_0 \left(1 + \frac{R^2}{a^2}\right)^{-1},$$

the mass inside radius R is

$$M(R) = \int_0^R 4\pi R^2 \rho(R)\, dR = 4\pi\rho_0 a^3 \left[\frac{R}{a} - \tan^{-1}\frac{R}{a}\right].$$

The squared circular orbital speed then is

$$V^2(R) = \frac{GM(R)}{R} = 4\pi G\rho_0 a^2 \left[1 - \frac{a}{R}\tan^{-1}\frac{R}{a}\right],$$

from which it follows that

$$VV' = \frac{2\pi G\rho_0 a^3}{R^2}\left[\tan^{-1}\frac{R}{a} - \frac{R/a}{1 + R^2/a^2}\right].$$

The expressions for A and B follow, but are messy and not particularly illuminating. Looking at the limit $R \gg a$, using the fact that $\tan^{-1} x \approx \frac{1}{2}\pi - 1/x$ for large x, and retaining only leading terms, we have

$$V \approx \sqrt{4\pi G\rho_0 a^2}\left[1 - \frac{\pi a}{4R}\right], \quad VV' \approx \frac{\pi^2 G\rho_0 a^3}{R^2}\left[1 - \frac{4a}{\pi R}\right],$$

so

$$A \approx -B \approx \frac{V}{2R} \approx \sqrt{\pi G\rho_0}\,\frac{a}{R}, \quad V' \approx \frac{\pi}{2}\sqrt{\pi G\rho_0}\,\frac{a^2}{R^2} \ll \frac{V}{R}.$$

4. If $V_0 = 220$ km/s, then the mass inside radius R is

$$M(R) = \frac{V_0^2 R}{G} = 1.13 \times 10^{10}\, R(\text{kpc})\, M_\odot.$$

From Problem 1, the light inside radius R is

$$\begin{aligned} L(R) &= 2L_* n_0\, h_z \int_0^R 2\pi R e^{-R/h_R} dR = 4\pi L_*\, n_0\, h_z\, h_R^2 \left[1 - \left(1 + \frac{R}{h_R}\right) e^{-R/h_R}\right] \\ &= L_G \left[1 - \left(1 + \frac{R}{h_R}\right) e^{-R/h_R}\right]. \end{aligned}$$

(a) For $R = R_0 = 8$ kpc, $M = 9.0 \times 10^{10}\, M_\odot$ and $L = 0.59\, L_G = 1.2 \times 10^{10}\, L_\odot$, so $M/L = 7.5\, M_\odot/L_\odot$.

(b) For $R = 10R_0 = 80$ kpc, $M = 9.0 \times 10^{11}\, M_\odot$ and $L = L_G = 2 \times 10^{10}\, L_\odot$, so $M/L = 45\, M_\odot/L_\odot$.

From Homework 1, Problem 5, with $dN/dM = AM^{-2.35}$ $L = (M/M_\odot)^4\, L_\odot$, $M_l = 0.2M_\odot$, and $M_u = 100M_\odot$, we find $M_{tot} = 4.48\, A(0.2\, M_\odot)^{-0.35}$, $L_{tot} = 0.378\, A\, L_\odot M_\odot^{-4}\, (100M_\odot)^{2.65}$. Thus $M_{tot}/L_{tot} = 5.9 \times 10^{-5}\, M_\odot/L_\odot$, much less than the observed value.

5. If the semimajor axis is a and the eccentricity is e, then the periastron distance is $r_p = a(1-e)$ and the apastron distance is $r_a = a(1+e)$. The orbital speeds at those points are $v_p = 4390$ km/s and $v_a = 920$ km/s, and the motion is transverse to the line from the star to the black hole. Thus, conservation of angular momentum implies $a(1-e)v_p = a(1+e)v_a$, so

$$\frac{1-e}{1+e} = \frac{v_a}{v_p},$$

so $e = 0.68$. Conservation of energy implies

$$\tfrac{1}{2}v_p^2 - \frac{GM}{a(1-e)} = \tfrac{1}{2}v_a^2 - \frac{GM}{a(1+e)},$$

so

$$\frac{GM}{a} = \frac{1-e^2}{4e}\left(v_p^2 - v_a^2\right).$$

Thus, in convenient units, $M(M_\odot)/a(AU) = 5020$. Finally, Kepler's third law implies that $P(yr)^2 = a(AU)^3/M(M_\odot)$ so we find $a = 815$ AU and $M = 4.1 \times 10^6 M_\odot$.