PHYS 431/531: Galactic Astrophysics

Fall 2021

Solutions to Homework #1

1. (a) The best estimate of the distance $d = 1/p$ is $d = 1/750 \times 10^{-6}$ pc $= 1333$ pc. A range in d can be obtained by using instead the upper (762×10^{-6} arcsec) and lower (738×10^{-6} arcsec) bounds on p: 1312 pc $< d <$ 1355 pc.

   (b) Using $m_G = 12.5$ and $d = 1333$ pc, we have $M_G = m_G - 5\log_{10} d(\mathrm{pc}) + 5$, so $M_G = 1.87$. A reasonable upper limit on M_G comes from taking the smallest value of d (1312 pc) and the largest value of m_G (12.51), giving $M_G = 1.92$. A lower limit comes from taking the largest value of d (1355 pc) and the largest smallest value of m_G (12.49), giving $M_G = 1.83$.

2. (a) Apparent magnitude m depends on luminosity L according to $m = -2.5\log_{10} L + \mathrm{const}$, where the constant includes both distance and calibration terms. For an unresolved binary of identical stars, the luminosity is $2L$, so the apparent magnitude is $m - 2.5\log_{10} 2 = m - 0.753$.

   (b) The star has $m_V = 12$ and, according to Sparke & Gallagher Table 1.4, the absolute visual magnitude of an A0 main sequence star is $M_V = 0.8$. Since $m - M = 5\log_{10}(d/10\,\mathrm{pc})$, it follows that the distance is $d = 1740$ pc.

   (c) Since $m_V - M_V = 5\log_{10} d(\mathrm{pc}) - 5$, it follows that $d = 10^{1+0.2(m_V - M_V)} = 72.4$ Mpc. Since the redshift is $z = 0.018$ the recession speed is $v = cz = 5400$ km/s. Using $v = H_0\, d$, we find $H_0 = 74.6$ km/s/Mpc, consistent with the currently accepted range.

3. (a) Stars of mass $0.5M_\odot$ have lifetimes of $\tau = 16$ Gyr, so all such stars are still around today.

   (b) Stars of mass $2M_\odot$ have lifetimes of $\tau = 1.25$ Gyr, so at an age of $t_0 = 10$ Gyr, all $2M_\odot$ stars born within the past 1.25 Gyr are still around today. Hence, the survivor fraction is

$$f = \frac{\int_{t_0-\tau}^{t_0} e^{-t/T} dt}{\int_0^{t_0} e^{-t/T} dt} = \frac{e^{\tau/T} - 1}{e^{t_0/T} - 1} = 1.9 \times 10^{-2}.$$

   (c) Stars of mass $4M_\odot$ have $\tau = 156$ Myr, so $f = 1.9 \times 10^{-3}$.

4. The binary center of mass lies at a distance of $d = 10$ pc.

   (a) The semi-major axes are $a_1 = \theta_1 d = 5$ AU and $a_2 = \theta_2 d = 20$ AU.

   (b) The semi-major axis of the relative orbit is $a = a_1 + a_2 = 25$ AU. Since $P^2 = a^3/M$ and $P = 90$ yr, it follows that $M = m_1 + m_2 = 25^3/90^2 = 1.93\,M_\odot$. Since $m_1/m_2 = a_2/a_1$, we find $m_1 = 1.54\,M_\odot, m_2 = 0.39\,M_\odot$.

5. (a) The stellar mass function is $dN/dm = Am^{-\alpha}$, for $0.2M_\odot = M_l < m < M_u = 100M_\odot$ and $\alpha = 2.35$. The number of stars having masses less than m is

$$N(m) = \int_{M_l}^{m} dN = \frac{A}{1-\alpha}\left[m^{1-\alpha}\right]_{M_l}^{m} = \frac{A}{\alpha - 1}\left(M_l^{1-\alpha} - m^{1-\alpha}\right).$$

The total number of stars is $N_{tot} = N(M_u) \approx AM_l^{1-\alpha}/(\alpha-1)$ for $M_l \ll M_u$, so the lower mass limit effectively determines the number of stars. Substituting for A we have

$$N(m) = N_{tot}\left[\frac{1-\left(\frac{m}{M_l}\right)^{1-\alpha}}{1-\left(\frac{M_u}{M_l}\right)^{1-\alpha}}\right],$$

which equals $\frac{1}{2}N_{tot}$ when $m = 1.7M_l = 0.33M_\odot$.

Similarly, the total mass in stars less massive than m is

$$M(m) = \int_{M_l}^{m} m dN = M_{tot}\left[\frac{1-\left(\frac{m}{M_l}\right)^{2-\alpha}}{1-\left(\frac{M_u}{M_l}\right)^{2-\alpha}}\right],$$

so the lower limit also dominates here (but barely, in this case—the second term in the denominator is roughly 0.11), and half the total mass is in stars having mass less than $M_1 = 5.33M_l = 1.07M_\odot$. Note that, if the second term in the denominator is neglected, the result becomes $M_1 = 7.25M_l = 1.45M_\odot$.

The total luminosity of stars less massive than m is

$$L(m) \propto \int_{M_l}^{m} m^4 dN = L_{tot}\left[\frac{1-\left(\frac{m}{M_l}\right)^{5-\alpha}}{1-\left(\frac{M_u}{M_l}\right)^{5-\alpha}}\right].$$

The total luminosity is $L_{tot} \approx AM_u^{5-\alpha}/(5-\alpha)$, so the massive stars dominate. Half of the total luminosity is accounted for by stars having masses greater than $M_2 = 0.77M_u = 77M_\odot$.

(b) Write the Kroupa mass function as

$$\xi(M) = \begin{cases} C\,M^{-\gamma} & (M < M_C) \\ B\,M^{-\beta} & (M_C < M < M_B) \\ A\,M^{-\alpha} & (M_B < M < M_U) \end{cases},$$

where $M_C = 0.1M_\odot$, $M_B = 0.5M_\odot$, $M_U = 100M_\odot$, $\gamma = 0.3$, $\beta = 1.3$, and $\alpha = 2.35$.

A is the same as before, and we can arbitrarily set it to 1 and work in solar masses throughout. The continuity of ξ at M_C and M_B implies

$$B = A\,M_B^{\beta-\alpha} = 2.071, \qquad C = B\,M_C^{\gamma-\beta} = 20.71.$$

The Kroupa and Salpeter distributions are identical for $M > M_B = 0.5$, so we can confine our attention to stars with $M < M_B$. For the Salpeter power-law, the number of stars with mass between M and M_B is

$$N_P(M) = \frac{1}{\alpha-1}\left[M^{1-\alpha} - M_B^{1-\alpha}\right] = 0.7407M^{-1.35} - 1.8882.$$

For the Kroupa distribution, the total number of stars with $M < M_B$ is

$$\begin{aligned} N_K &= \int_0^{M_C} CM^{-\gamma}dM + \int_{M_C}^{M_B} BM^{-\beta}dM \\ &= \frac{C}{1-\gamma}M_C^{1-\gamma} + \frac{B}{\beta-1}\left[M_C^{1-\beta} - M_B^{1-\beta}\right] \\ &= 11.178. \end{aligned}$$

Hence the power-law and Kroupa results are the same for $M = 0.119\, M_\odot$.

Similarly, for the total mass,

$$M_P(M) = \frac{1}{\alpha - 2}\left[M^{2-\alpha} - M_B^{2-\alpha}\right] = 2.857\, M^{-0.35} - 3.641,$$

while

$$\begin{aligned} M_K &= \int_0^{M_C} C M^{1-\gamma} dM + \int_{M_C}^{M_B} B M^{1-\beta} dM \\ &= \frac{C}{2-\gamma} M_C^{2-\gamma} + \frac{B}{2-\beta}\left[M_B^{2-\beta} - M_C^{2-\beta}\right] \\ &= 1.474, \end{aligned}$$

so the power-law and Kroupa results agree for $M = 0.189\, M_\odot$.

Finally, for the total luminosity,

$$L_P(M) = \frac{1}{5 - \alpha}\left[M_B^{5-\alpha} - M^{5-\alpha}\right] = 0.06012 - 0.3774 M^{2.65},$$

while

$$\begin{aligned} L_K &= \int_0^{M_C} C M^{4-\gamma} dM + \int_{M_C}^{M_B} B M^{4-\beta} dM \\ &= \frac{C}{5-\gamma} M_C^{5-\gamma} + \frac{B}{5-\beta}\left[M_B^{5-\beta} - M_C^{5-\beta}\right] \\ &= 0.04305, \end{aligned}$$

so the power-law and Kroupa results agree for $M = 0.311\, M_\odot$.