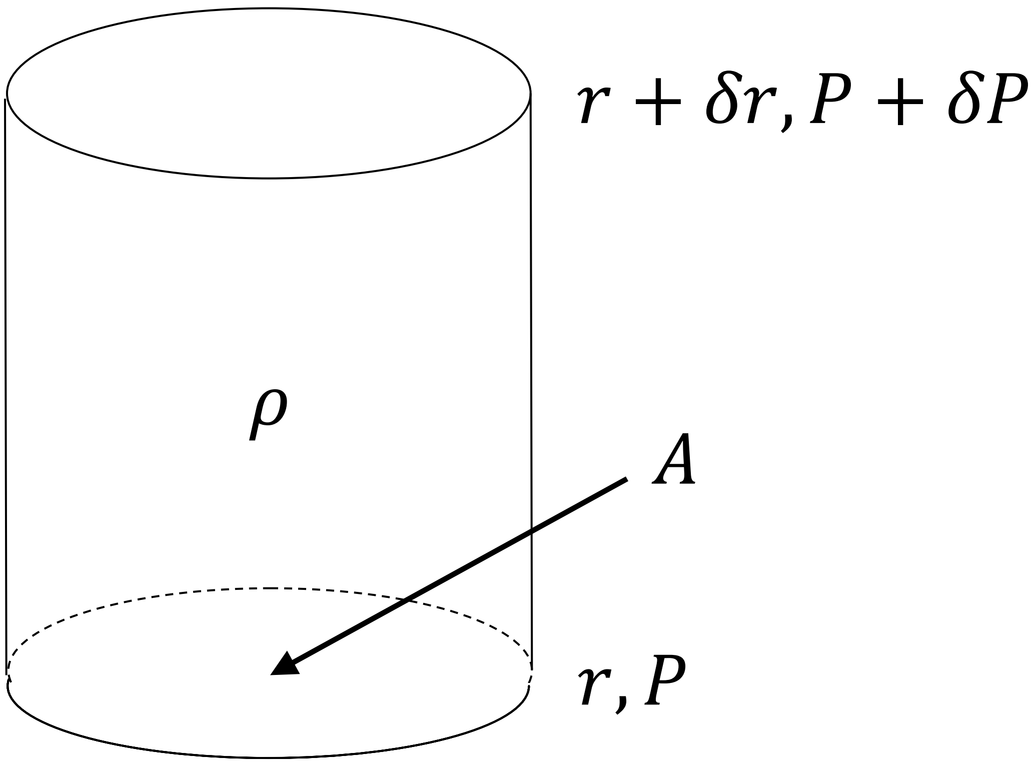
Hydrostatic Equilibrium



$$\delta m = A \delta r \, \rho$$

$$F_p = -A \delta P \quad \uparrow$$

$$F_g = \frac{G M(r) \delta m}{r^2} \quad \downarrow$$

$$\Rightarrow \frac{dP}{dr} = -\frac{G M(r) \rho(r)}{r^2}$$

Virial Theorem

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = -\int_0^R 4\pi r^3 \frac{GM(r)\rho(r)}{r^2} dr$$

$$[4\pi r^3 P]_0^R - 3\int_0^R 4\pi r^2 P(r)\, dr$$

$$= -3\int_0^R 4\pi r^2 P(r)\, dr$$

$$-\int_0^R \frac{GM(r)\, dm}{r}$$

$$= E_{gr}$$

Virial Theorem

Ideal, non-relativistic gas:

$$u = \frac{3}{2} nkT$$

$$P = nkT = \frac{2}{3} u$$

$$\Longrightarrow \; -3 \int_0^R 4\pi r^2 P(r)\, dr = -2 \int_0^R 4\pi r^2 u(r)\, dr$$

$$= -2\, E_{th} = E_{gr}$$

$$\boxed{2\, E_{th} + E_{gr} = 0}$$

Virial Theorem

$$2\,E_{th} + E_{gr} = 0$$

$$\Longrightarrow\; E_{tot} = E_{th} + E_{gr} = -E_{th} < 0$$

star emits radiation $\Longrightarrow\; E_{tot}\downarrow \;\Longrightarrow\; E_{th}\uparrow$

negative specific heat!

Virial Temperature

$$E_{gr} \approx \frac{1}{2}\frac{GM^2}{R}$$

$$E_{th} = \frac{3}{2} N\, k\, T_{vir} \qquad \left(N = \frac{M}{\mu}\right)$$

pure ionized hydrogen, $\mu \approx \frac{1}{2}\, m_H$

$\Longrightarrow\ T_{vir} \approx 4\times10^6\ K$ for the Sun

Central Solar Pressure

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4}$$

$$\Longrightarrow P_c \approx \int_0^{M_\odot} \frac{GM(r)}{4\pi r^4}\, dM > \int_0^{M_\odot} \frac{GM(r)}{4\pi {R_\odot}^4}\, dM = 4.5\times 10^{13}\ \mathrm{Pa}$$

$$= 4.5\times 10^{8}\ \mathrm{atm}$$

Gas Composition

H fraction by mass $X = \frac{\rho_H}{\rho_{tot}}$

He fraction by mass $Y = \frac{\rho_{He}}{\rho_{tot}}$

"other" fraction by mass $Z = \frac{\rho_A}{\rho_{tot}}$

obviously $X + Y + Z = 1$

deep in the stellar interior, assume fully ionized, ideal, non-relativistic gas

want to know (1) mean particle mass, (2) equation of state

Mean Particle Mass

mean mass $\quad \mu = \frac{\sum_i n_i m_i}{\sum_i n_i} = \frac{\rho}{n}$

fully ionized: $H \longrightarrow p + e$ 2 particles, mass 1 m_H

$He \longrightarrow \alpha + 2e$ 3 particles, mass 4 m_H

$A \longrightarrow A_n + \frac{1}{2} Ae$ $1 + \frac{\text{A}}{2}$ particles, mass A m_H

Periodic Table of the Elements

1 1IA 1A	2 IIA 2A	3 IIIB 3B	4 IVB 4B	5 VB 5B	6 VIB 6B	7 VIIB 7B	8	9 VIII 8	10	11 IB 1B	12 IIB 2B	13 IIIA 3A	14 IVA 4A	15 VA 5A	16 VIA 6A	17 VIIA 7A	18 VIIIA 8A
1 H Hydrogen 1.008																	2 He Helium 4.003
3 Li Lithium 6.941	4 Be Beryllium 9.012											5 B Boron 10.811	6 C Carbon 12.011	7 N Nitrogen 14.007	8 O Oxygen 15.999	9 F Fluorine 18.998	10 Ne Neon 20.180
11 Na Sodium 22.99	12 Mg Magnesium 24.305											13 Al Aluminum 26.982	14 Si Silicon 28.086	15 P Phosphorus 30.974	16 S Sulfur 32.066	17 Cl Chlorine 35.453	18 Ar Argon 39.948
19 K Potassium 39.098	20 Ca Calcium 40.078	21 Sc Scandium 44.956	22 Ti Titanium 47.867	23 V Vanadium 50.942	24 Cr Chromium 51.996	25 Mn Manganese 54.938	26 Fe Iron 55.845	27 Co Cobalt 58.933	28 Ni Nickel 58.693	29 Cu Copper 63.546	30 Zn Zinc 65.38	31 Ga Gallium 69.723	32 Ge Germanium 72.631	33 As Arsenic 74.922	34 Se Selenium 78.971	35 Br Bromine 79.904	36 Kr Krypton 83.789
37 Rb Rubidium 85.468	38 Sr Strontium 87.62	39 Y Yttrium 88.906	40 Zr Zirconium 91.224	41 Nb Niobium 92.906	42 Mo Molybdenum 95.95	43 Tc Technetium 98.907	44 Ru Ruthenium 101.07	45 Rh Rhodium 102.906	46 Pd Palladium 106.42	47 Ag Silver 107.868	48 Cd Cadmium 112.414	49 In Indium 114.818	50 Sn Tin 118.711	51 Sb Antimony 121.760	52 Te Tellurium 127.6	53 I Iodine 126.904	54 Xe Xenon 131.294
55 Cs Cesium 132.905	56 Ba Barium 137.328	57-71	72 Hf Hafnium 178.49	73 Ta Tantalum 180.948	74 W Tungsten 183.84	75 Re Rhenium 186.207	76 Os Osmium 190.23	77 Ir Iridium 192.217	78 Pt Platinum 195.085	79 Au Gold 196.967	80 Hg Mercury 200.592	81 Tl Thallium 204.383	82 Pb Lead 207.2	83 Bi Bismuth 208.980	84 Po Polonium [208.982]	85 At Astatine 209.987	86 Rn Radon 222.018
87 Fr Francium 223.020	88 Ra Radium 226.025	89-103	104 Rf Rutherfordium [261]	105 Db Dubnium [262]	106 Sg Seaborgium [266]	107 Bh Bohrium [264]	108 Hs Hassium [269]	109 Mt Meitnerium [278]	110 Ds Darmstadtium [281]	111 Rg Roentgenium [280]	112 Cn Copernicium [285]	113 Nh Nihonium [286]	114 Fl Flerovium [289]	115 Mc Moscovium [286]	116 Lv Livermorium [293]	117 Ts Tennessine [294]	118 Og Oganesson [294]

Lanthanide Series	57 La Lanthanum 138.905	58 Ce Cerium 140.116	59 Pr Praseodymium 140.908	60 Nd Neodymium 144.243	61 Pm Promethium 144.913	62 Sm Samarium 150.36	63 Eu Europium 151.964	64 Gd Gadolinium 157.25	65 Tb Terbium 158.925	66 Dy Dysprosium 162.500	67 Ho Holmium 164.930	68 Er Erbium 167.259	69 Tm Thulium 168.934	70 Yb Ytterbium 173.055	71 Lu Lutetium 174.967
Actinide Series	89 Ac Actinium 227.028	90 Th Thorium 232.038	91 Pa Protactinium 231.036	92 U Uranium 238.029	93 Np Neptunium 237.048	94 Pu Plutonium 244.064	95 Am Americium 243.061	96 Cm Curium 247.070	97 Bk Berkelium 247.070	98 Cf Californium 251.080	99 Es Einsteinium [254]	100 Fm Fermium 257.095	101 Md Mendelevium 258.1	102 No Nobelium 259.101	103 Lr Lawrencium [262]

Alkali Metal | Alkaline Earth | Transition Metal | Basic Metal | Semimetal | Nonmetal | Halogen | Noble Gas | Lanthanide | Actinide

Mean Particle Mass

mean mass $\mu = \frac{\sum_i n_i m_i}{\sum_i n_i} = \frac{\rho}{n}$

fully ionized: $H \longrightarrow p + e$ 2 particles, mass 1 m_H

$He \longrightarrow \alpha + 2e$ 3 particles, mass 4 m_H

$A \longrightarrow A_n + \frac{1}{2} Ae$ $1 + \frac{\mathrm{A}}{2}$ particles, mass $A\, m_H$

unit volume: $n_H = \frac{X\rho}{m_H}$ 2 particles

$n_{He} = \frac{Y\rho}{4m_H}$ 3 particles

$n_A = \frac{Z\rho}{Am_H}$ $1 + \frac{A}{2}$ particles

Mean Particle Mass

number density

$$n = 2n_H + 3n_{He} + \frac{A}{2}n_A$$
$$= \frac{\rho}{m_H}\left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)$$
$$= \frac{\rho}{2m_H}\left(1 + 3X + \frac{1}{2}Y\right)$$

mean mass

$$\mu = \frac{\rho}{n} = \frac{2m_H}{1 + 3X + \frac{1}{2}Y}$$

Examples

Sun at birth $X = 0.71,\ Y = 0.27,\ Z = 0.02$

$$\Longrightarrow \quad \mu = \ 0.618\, m_H$$

solar core $X = 0.34,\ Y = 0.64,\ Z = 0.02$

$$\Longrightarrow \quad \mu = \ 0.855\, m_H$$

Sun's outer layers $X = 0.75,\ Y = 0.24,\ Z = 0.01$

$$\Longrightarrow \quad \mu = \ 0.593\, m_H$$

Equation of State

pressure

$$P = \; n\,k\,T \; + \; P_{rad}$$

$$= \; \frac{\rho k T}{2m_H}\left(1 + 3X + \frac{1}{2}Y\right) + \frac{1}{3}aT^3$$

Equations of Stellar Structure

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}$$ P, M, ρ

$$\frac{dM}{dr} = 4\pi r^2 \rho$$ M, ρ

$$\frac{dL}{dr} = 4\pi r^2 \rho \left(\epsilon - \epsilon_\nu\right)$$ L, ρ

equation of state: $P(\rho, T, X, Y)$

Radiation Transfer

$$\frac{dI_\nu}{dx} = -\alpha_\nu I_\nu + j_\nu$$

Absorption only:

$$\frac{dI_\nu}{dx} = -\alpha_\nu I_\nu$$

$\tau = 1$

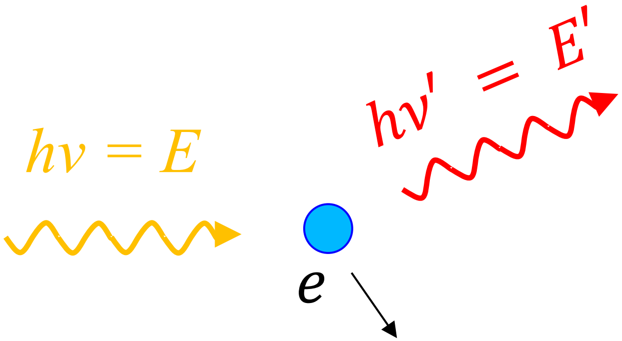
$\Rightarrow \quad x =$ mean free path

$l \sim 1/\alpha$

$$\Rightarrow \quad I_\nu = I_{\nu,0}\, e^{-\tau_\nu}$$

where $\tau_\nu = \int \alpha_\nu \, dx$ = optical depth

Electron Scattering



Thompson scattering cross section

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$$

$$= 6.7\times10^{-25}\ \text{cm}^2$$

Mean free path in Sun due to electron scattering is

$$l_{es} = \frac{1}{n_e \sigma_T} \approx \frac{m_H}{\rho \sigma_T} \approx 1\ \text{mm} \ll R_\odot = 7\times10^{10}\ \text{cm}$$

solar interior is <u>opaque</u> to electromagnetic radiation

Radiative Energy Transport

photons random walk though the solar interior

expected distance traveled after N "hops" is

$$\Delta r \approx N^{1/2} l_{es}$$

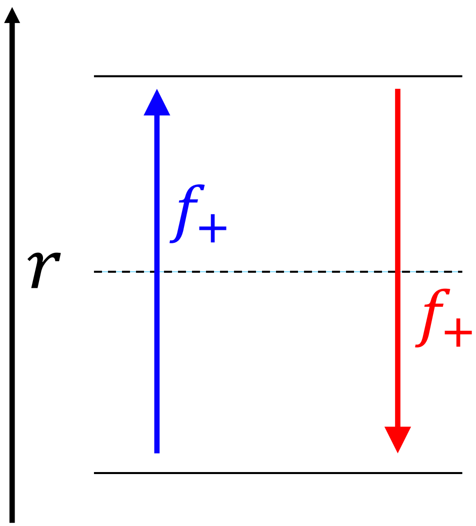
time taken is

$$\Delta t \approx \frac{N^{1/2} l_{es}}{c}$$

for $\Delta r \sim R_{\odot}$ and $l_{es} = 1$ mm

$$\Longrightarrow \mathrm{N} \sim 5{\times}10^{23}, \qquad \Delta t \sim 50{,}000 \;\; \mathrm{yr}$$

Radiative Energy Transport



$$r + l;\ \ u + \delta u;\ \ \delta u = -l\frac{du}{dr};\ \ u = aT^4$$

upward flux of particles is $f_+ = \frac{1}{6} n\bar{v}$

upward net energy flux is $F_+ = \frac{1}{6} n\bar{v}\delta u$

r, u

similarly, downward net energy flux is $F_- = \frac{1}{6} n\bar{v}\delta u$

total flux $F = -\frac{1}{3} n\bar{v}\, l\, \frac{du}{dr} = -\frac{4}{3} n\bar{v}\, l\, T^3 \frac{dT}{dr} = \frac{L}{4\pi r^2}$

$$\Longrightarrow\ \ \frac{dT}{dr} = -\frac{3L\kappa\rho}{16\pi a c r^2 T^3}$$

Equations of Stellar Structure

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}$$ P, M, ρ

$$\frac{dM}{dr} = 4\pi r^2 \rho$$ M, ρ

$$\frac{dL}{dr} = 4\pi r^2 \rho \, (\epsilon - \epsilon_\nu)$$ L, ρ

$$\frac{dT}{dr} = -\frac{3\kappa L\rho}{16\pi ac \, r^2 \, T^3}$$ T, L, ρ

equation of state: $P(\rho, T, X, Y)$

Opacity

$$\kappa_{es} = 0.02\,(1+X)\ [\mathrm{m}^2/\mathrm{kg}]$$

Opacity Mechanisms

electron scattering

bound-bound	=	line formation
bound-free	=	ionization
free-free	=	bremsstrahlung

Opacity

$$\kappa_{es} = 0.02\,(1+X)\ [\mathrm{m}^2/\mathrm{kg}]$$

$$\kappa_{ff} = 1.2 \times 10^4\,(1-Z)(1+X)\left(\frac{\rho}{10^3\ \mathrm{kg/m^3}}\right)\left(\frac{T}{10^5\ \mathrm{K}}\right)^{-7/2}$$

$$\kappa_{bf} = 1.4 \times 10^4\,Z(1+X)\left(\frac{\rho}{10^3\ \mathrm{kg/m^3}}\right)\left(\frac{T}{10^5\ \mathrm{K}}\right)^{-7/2}$$

Kramers Law: $\kappa \propto \rho\, T^{-7/2}$