Physics 115: Contemporary Physics III

Spring 2013

Homework 5 Solutions

1. In the unprimed frame, the charged particle is at rest and the fields are

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}, \qquad \mathbf{B} = 0.$$

Transforming to the primed frame moving with velocity $\mathbf{v}$ relative to the first, and noting that $\epsilon_0\mu_0 = 1/c^2$, we have

$$B'_{\parallel} = 0, \qquad B'_{\perp} = -\gamma\frac{\mathbf{v}\times\mathbf{E}}{c^2} \;=\; -\gamma\,\frac{\mu_0}{4\pi}\,\frac{q\mathbf{v}\times\hat{\mathbf{r}}}{r^2},$$

which (apart from the γ) is the Biot-Savart law for the magnetic field due to a particle with velocity $-\mathbf{v}$.

4. First, argue from considerations of symmetry that the gravitational field $\mathbf{g}$ at location $\mathbf{r}$ relative to the mass m must be radial (no transverse component) and depend only on $r = |\mathbf{r}|$ (no angular dependence), i.e. $\mathbf{g} = g(r)\,\hat{\mathbf{r}}$. Second, surround the mass with a Gaussian surface in the form of a sphere centered on it. Then $d\mathbf{A} = dA\,\hat{\mathbf{r}}$ and $\mathbf{g}\cdot d\mathbf{A} = g(r)\,dA$, so $\Phi_g = 4\pi r^2 g(r) = -4\pi Gm$, by Gauss's law. Hence

$$g(r) = -\frac{GM}{r^2}.$$

The significance of the minus sign is that it means the gravitational field is attractive (for positive mass m).

$$\begin{aligned}\Phi_{net} &= \frac{\Sigma q_{inside}}{\epsilon_0} \\ \Sigma q_{inside} &= (224\ \frac{\mathrm{N\cdot m^2}}{\mathrm{C}})(8.85\times10^{-12}\ \frac{\mathrm{N\cdot C^2}}{\mathrm{m^2}}) \\ &= 1.98\times10^{-9}\ \mathrm{C}\end{aligned}$$

(c) This pattern of electric field is approximately created by a uniformly charged capacitor with one plate aligned with the diagonal of the cube. If the plate separation is at least the length of $\frac{1}{2}(0.55\ \mathrm{m})$ and if the area of each plate is much larger than the cross-sectional area of the cube, sliced along the diagonal, then the approximation is valid since $\vec{E}_{outside} \approx 0$ and $\vec{E}_{inside} \approx$ uniform.

22.X.20

For whatever Gaussian surface you choose, the electric field due to the ring at the surface will not be uniform in magnitude or direction across the entire surface. As a result, there is no symmetry that can be used to simplify the integral. Gauss' law is still valid; however, the integral cannot be easily solved.

22.X.21

According to Gauss' law, electric field is zero in the interior of a metal, even if there is a hole in the metal and the metal is simply a shell. This phenomenon can be referred to as "screening." Thus, if your car is struck by lightening, the electric field within the car will remain zero even if the car acquires a net charge.

If you step out of the car, then excess charge on the car can flow from the car through you to ground.

22.P.22

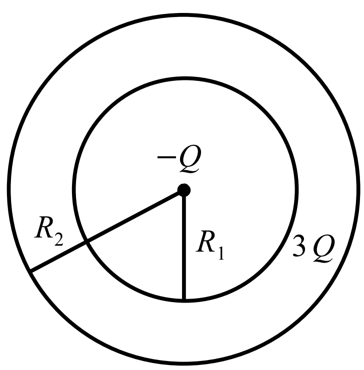

Figure 2: A sketch of the situation

(a) For $r < R_1$, the electric field is

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

$$E = \frac{1}{4\pi\epsilon_0}\left(\frac{-Q + 3Q\frac{(r^3 - R_1^3)}{(R_2^3 - R_1^3)}}{r^2}\right)$$

$\vec{E}$ is in the radial direction with the magnitude shown above for $R_1 < r < R_2$.

(c) For $R_2 < r$, $\Sigma q_{\text{inside}} = -Q + 3Q = 2Q$. So Gauss' Law gives

$$\begin{aligned} E &= \frac{2Q}{4\pi r^3 \epsilon_0} \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{2Q}{r^2}\right) \end{aligned}$$

$\vec{E}$ is radial, of course.

22.P.23

(a) In this part of the problem we reason from what we know about the field to determine what and where the charge must be. To see if there is any charge inside the wire, we draw a (mathematical, imaginary) cylinder of length d completely inside the inner wire, and far from the ends of the wire, as shown in Figure 3.

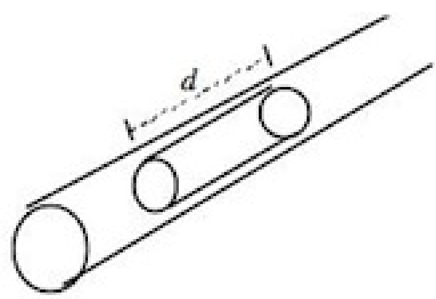

Figure 3: Gaussian cylinder inside the wire, in Question 22.P.23.

By Gauss's Law:

$$\oint \vec{E} \cdot \hat{n} dA = \frac{\Sigma q_{inside}}{\varepsilon_0}$$

At equilibrium E $= 0$ in metal, so flux on cylinder $= 0$ and $\Sigma q_{inside} = 0$.

So there cannot be any charge inside the wire. This means that all the positive charge is on the outer surface of the inner wire. To see how much of the negative charge is on the inner surface of the outer cylinder, and how much is on the outer surface, we draw a (mathematical) cylindrical surface inside the metal of the outer cylinder, again far from the ends of the wire, as shown in Figure 4.

Apply Gauss's Law:

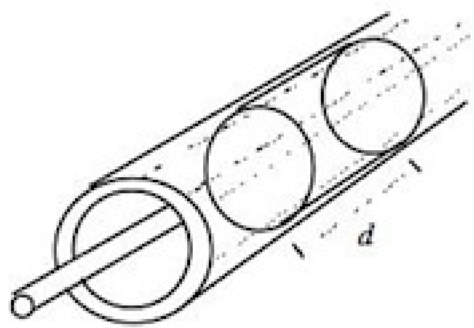

Figure 4: Gaussian cylinder inside the outer cylinder, in Question 22.P.23.

$$\begin{aligned}\oint \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \\ \int_{\text{end caps}} \vec{\mathrm{E}} \cdot \hat{n} dA + \int_{\text{curved sides}} \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0}\end{aligned}$$

The curved (side) part of the cylinder is entirely within the metal, and as before $E = 0$ inside the metal in equilibrium, so flux on the curved part is zero. Only part of the end cap is inside the metal; the other part is in the air gap, where there is a nonzero electric field.

Looking end-on at the cylinder, we see that $\hat{n}$ is out of the page, while $\vec{\mathrm{E}}$ points out from the inner wire (by symmetry) and is in the plane of the page, parallel to the surface of the end cap, as shown in Figure 5.

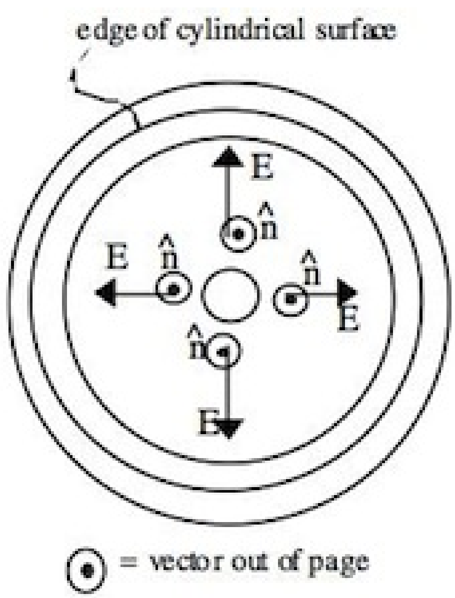

Figure 5: Electric field and the unit normal vector shown in the end view, in Question 22.P.23.

The field is therefore perpendicular to $\hat{n}$, and $\vec{\mathrm{E}} \cdot \hat{n} = 0$. So the flux on the end caps of the mathematical cylinder is zero, and therefore the net flux on the cylinder is zero. By Gauss's Law, the net charge enclosed by this cylindrical surface must therefore be 0. We know that the cylinder encloses a charge $(+Q/L)d$ on the inner wire. Thus:

$$\begin{aligned} 0 &= \left(\frac{Q}{L}\right) d + (\text{density on surface}) d \\ (\text{density on surface}) &= -\left(\frac{Q}{L}\right) \end{aligned}$$

All the negative charge is on the inner surface.

(b) In this part of the problem we reason from what we now know about the charge distribution, plus what we know about the direction and symmetry of E , to get an algebraic expression for the magnitude of E inside the air gap. We place our mathematical surface so that at least one side (the curved part) is located in the region where we want to know the electric field—-in the air gap (see Figure 6).

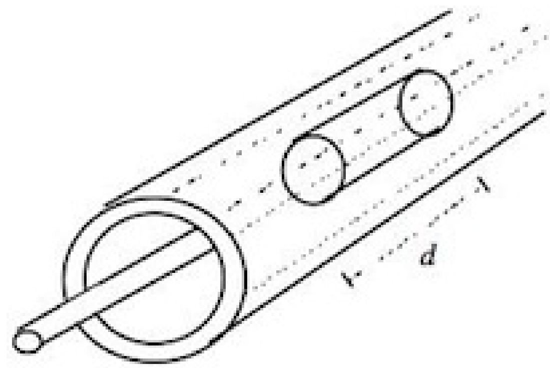

Figure 6: Gaussian surface for part(b) in Question 22.P.23.

Again:

$$\begin{aligned} \oint \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \\ \int_{\text{end caps}} \vec{\mathrm{E}} \cdot \hat{n} dA + \int_{\text{curved sides}} \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \end{aligned}$$

By symmetry, the electric field in the air gap must point outward from the wire.

End Caps: On the end caps, therefore, as before, $\vec{\mathrm{E}} \cdot \hat{n} = 0$ and the flux=0. See Figure 7.

Curved Surface: E is nonzero, and points outward, parallel to $\hat{n}$ as shown in Figure 8.

So, $\vec{\mathrm{E}} \cdot \hat{n} = E$. On the curved surface, at a constant distance from the wire, the magnitude E is constant, by symmetry. We can partially evaluate the flux:

$$\begin{aligned} \int_{\text{curved sides}} \vec{\mathrm{E}} \cdot \hat{n} dA &= \int_{\text{curved sides}} E cos(0) dA \\ &= E \int_{closed\ surface} dA \\ &= E 2\pi r d \end{aligned}$$

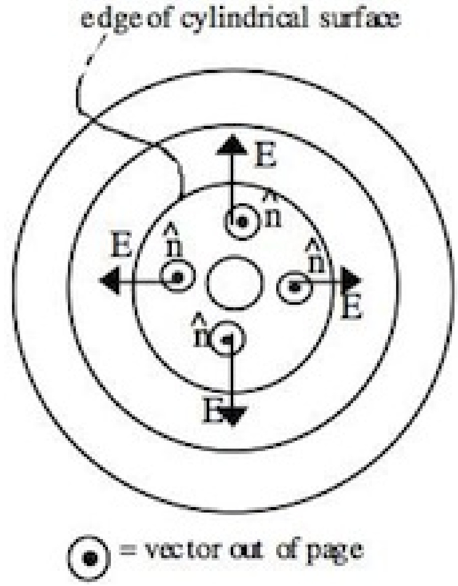

Figure 7: End caps for part(b) in Question 22.P.23.

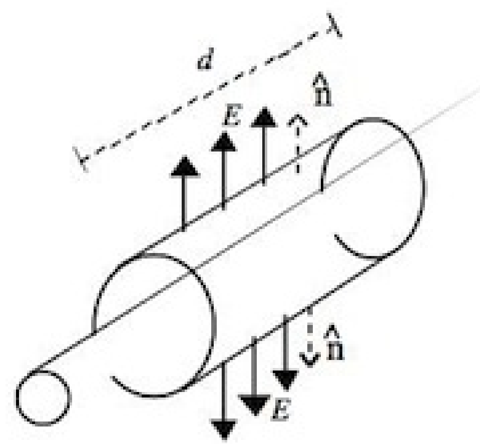

Figure 8: Electric field and unit normal vector for part through the curved surface for part (b) in Question 22.P.23.

(where $2\pi rd$ is the area of the side of the cylinder)

The charge inside the cylinder is equal to (charge per unit length)? (length of cylinder):

$$\begin{aligned}
\oint \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \\
\int_{\text{end caps}} \vec{\mathrm{E}} \cdot \hat{n} dA + \int_{\text{curved sides}} \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \\
0 + E 2\pi r d &= \frac{\left(\frac{+Q}{L}\right) d}{\varepsilon_0} \\
E &= \frac{\left(\frac{+Q}{L}\right) d}{2\pi r d \varepsilon_0} \\
&= \frac{Q/L}{2\pi \varepsilon_0 r}
\end{aligned}$$

This is the magnitude of E in the air gap. Note that it is the same as the electric field of a uniformly charged long straight wire.

(c) Finally we find the magnitude of the electric field outside the whole assembly. Our closed surface now extends around both wires as shown in Figure 9.

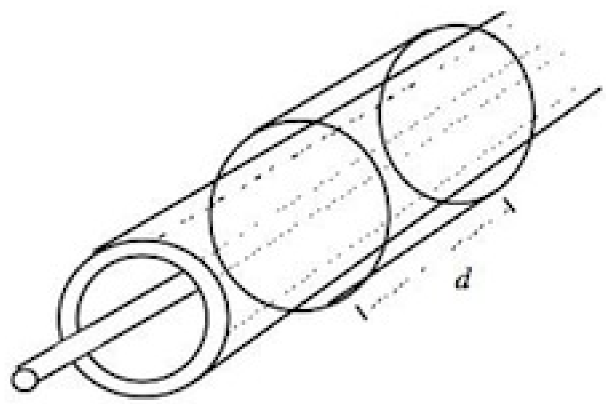

Figure 9: Gaussian surface for part (c) in Question 22.P.23.

As above, on end caps $\vec{\mathrm{E}} \cdot \hat{n} = 0$. On curved surface E must point outward, so $\vec{\mathrm{E}} \cdot \hat{n} = E$, so flux $= E(2\pi r d)$. Gauss' Law gives:

$$\begin{aligned}
\oint \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \\
\int_{\text{end caps}} \vec{\mathrm{E}} \cdot \hat{n} dA + \int_{\text{curved sides}} \vec{\mathrm{E}} \cdot \hat{n} dA &= \frac{\Sigma q_{inside}}{\varepsilon_0} \\
0 + E 2\pi r d &= \frac{\left(\frac{+Q}{L}\right) d + \left(\frac{-Q}{L}\right) d}{\varepsilon_0} \\
E 2\pi r d &= 0 \\
E &= 0
\end{aligned}$$

Therefore, $E = 0$ outside of the coaxial cable.

22.P.24

$$\begin{aligned}
\oint \vec{B}\cdot d\vec{l} &= \vec{B}_{top}\cdot d\vec{l}_{top} + \vec{B}_{right}\cdot d\vec{l}_{right} + \vec{B}_{bottom}\cdot d\vec{l}_{bottom} + \vec{B}_{left}\cdot d\vec{l}_{left} \\
&= (1.6\times 10^{-4}\ \mathrm{T})\cos(35^\circ)(0.6\ \mathrm{m}) + 0 + (1.6\times 10^{-4}\ \mathrm{T})\cos(35^\circ)(0.6\ \mathrm{m}) + 0 \\
&= 1.57\times 10^{-4}\ \mathrm{T}
\end{aligned}$$

Ampere's Law is

$$\begin{aligned}
\oint \vec{B}\cdot d\vec{l} &= \mu_0 I \\
I &= \frac{1.57\times 10^{-4}\ \mathrm{T}}{\mu_0} \\
&= 125\ \mathrm{A}
\end{aligned}$$

22.P.30

Choose a D-shaped Amperian path along the center line of the plastic frame (the dashed gray path in Figure 10).

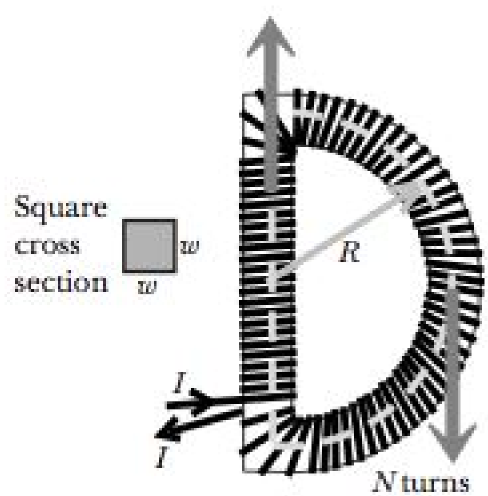

Figure 10: Amperian path and a few magnetic field vectors for Question 22.P.30.

Ampere's law is

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{inside}$$

We are told that the magnetic field has approximately the same magnitude B throughout the plastic, and presumably it is approximately parallel to our chosen path. Therefore $\vec{B}\cdot d\vec{l} = Bdl$, and B can be taken out of the integral.

Also, a soap bubble stretched over our path is pierced N times by wires carrying current I, so we have

$$B\oint dl = \mu_0 NI$$

The integral of dl is the path length, which is one diameter plus one-half of the circumference or $(2R+\pi R)$. Substitute into Ampere's law and solve for B.

$$\begin{aligned} B\oint dl &= \mu_0 NI \\ B(2R+\pi R) &= \mu_0 NI \\ B &= \frac{\mu_0 NI}{(2R+\pi R)} \end{aligned}$$

22.P.31

(a) From the diagram in Figure 11 it is clear that there is cancellation of the vertical components of magnetic field contributed by two wires to the left and the right of the observation location. Therefore the direction of the magnetic field must be to the left at the location above the wires and to the right at the location below the wires.

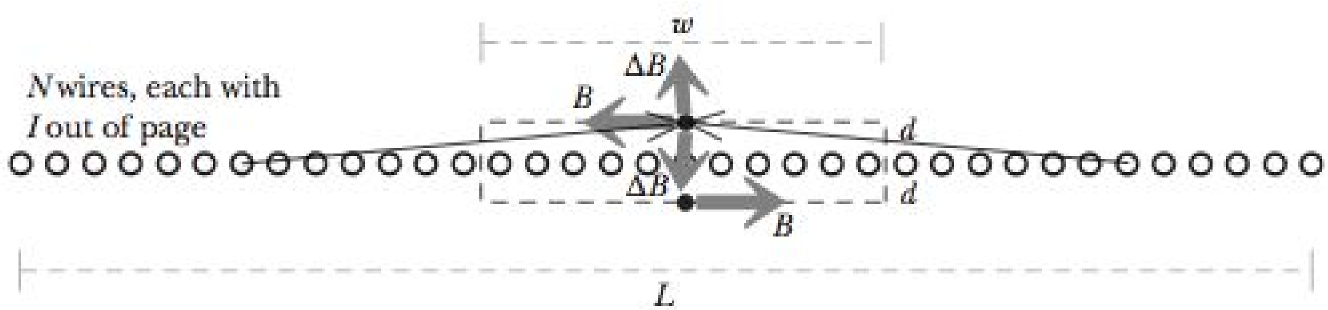

Figure 11: Magnetic field at the given locations in Question 22.P.31.

(b) Use AmpereÕs law, and go counterclockwise around the closed rectangular path.

Along the sides of the path $\int \vec{\mathrm{B}}\cdot d\vec{\mathrm{l}} = 0$, since $\vec{\mathrm{B}}$ is perpendicular to $d\vec{\mathrm{l}}$.

Along the upper part of the path, $\int \vec{\mathrm{B}}\cdot d\vec{\mathrm{l}} = Bw$.

Along the lower part of the path, $\int \vec{\mathrm{B}}\cdot d\vec{\mathrm{l}} = Bw$.

Therefore, applying Ampere's law gives

$$\begin{aligned} \int \vec{\mathrm{B}}\cdot d\vec{\mathrm{l}} &= \mu_0 I_{\text{inside path}} \\ 2Bw &= \mu_0 I_{\text{inside path}} \end{aligned}$$

The current inside the amperian loop is $I_{\text{inside path}} = \left(\frac{N}{L}\right) wI$ since there are N/L current-carrying wires per meter, and a width w of the enclosing path. Thus,

$$B = \frac{\mu_0}{4\pi}\frac{2(I_2 - I_1)}{R}$$

This is the same as the magnitude of the magnetic field at a distance R from a long, straight wire with current $I_2 - I_1$. Note that if $I_1 = I_2$, then $B = 0$ as expected. Also, if $I_1 > I_2$, then B will be negative meaning that would be tangent to the path and clockwise.

22.P.34

The current density (current/area) is the same throughout the wire. Sketch an Amperian loop within the wire with radius r. Then, the current through this loop i per unit area is the same as the total current per unit area.

$$\begin{aligned} \frac{i}{\pi r^2} &= \frac{I}{\pi R^2} \\ i &= I\frac{r^2}{R^2} \end{aligned}$$

The magnetic field is tangential to the Amperian loop. Apply Ampere's law by integrating counterclockwise around the loop if looking at the loop from the right end. Then the current flows out of the surface and is positive.

$$\begin{aligned} \oint \vec{\mathrm{B}} \cdot d\vec{l} &= \mu_0 I_{\text{inside path}} \\ B(2\pi R) &= \mu_0 I\frac{r^2}{R^2} \\ B &= \frac{\mu_0}{2\pi}\frac{Ir^2}{R^3} \end{aligned}$$

22.X.35

Units of div($\vec{E}$) are $\frac{\mathrm{N/C\cdot m^2}}{\mathrm{m^3}} = \frac{\mathrm{N}}{(\mathrm{C\cdot m})} = \mathrm{N/C/m}$.

22.X.36

Treat the nucleus as a sphere with uniform charge distribution throughout the volume of the sphere. The electric field at the surface is radial and has a magnitude equal to that of a point charge Q at the center of the sphere. The area of the sphere is $4\pi R^2$ and its volume is $4/3\pi R^3$.

The divergence of the electric field through the sphere is

$$\begin{aligned} \mathrm{div}(\vec{\mathrm{E}}) &= \frac{\oint \vec{\mathrm{E}}\cdot\hat{n}dA}{\Delta V} \\ &= \frac{E(4\pi R^2)}{4/3\pi R^3} \\ &= \frac{3E}{R} \end{aligned}$$