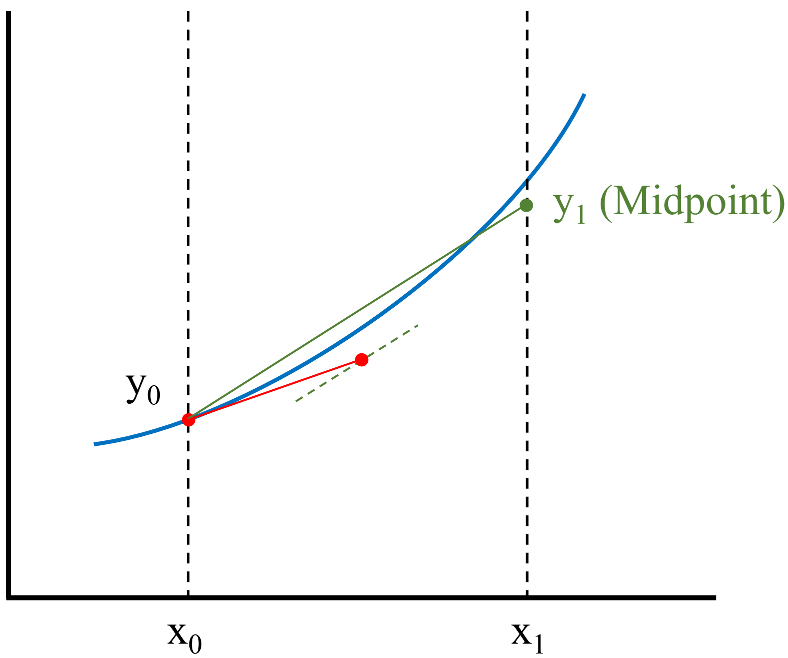
The Mid-point Method

The Euler method uses a one-sided estimate of the derivative to advance the system from state i to state $i+1$. We would do better if we could use a centered estimate—that is, if we knew the derivative at the center of the interval, at time "$i+\frac{1}{2}$". The reason for this becomes clear if we look at the Taylor series. Imagine applying the Euler method, but use the derivative at the mid-point, instead of at the start of the range. We obtain

$$\begin{aligned}\tilde{y} &= y(x) + \delta x\, f(x + \tfrac{1}{2}\delta x) \\ &= y(x) + \delta x \left[f(x) + \tfrac{1}{2}\delta x f'(x) + O(\delta x^2) \right] \\ &= y(x) + \delta x\, f(x) + \tfrac{1}{2}\delta x^2 f'(x) + O(\delta x^3)\end{aligned}$$

which is the correct second-order expression. We have gained an extra order in accuracy just by evaluating the derivative at a different location! Once again, it is easiest to see what's going on geometrically:



How can we determine the derivative at the mid-point of the interval? We use Euler! The point is that, even though Euler is not particularly accurate, when this inaccuracy occurs in the derivative (which gets multiplied by the timestep), the result is "accurate enough" to produce an overall gain. This double application of the Euler scheme to refine the net accuracy is called the Mid-point method. More formally, we can define the scheme as follows:

$$\begin{aligned}\delta y &= \delta x\, f(x_i, y_i) \\ y_{i+1} &= y_i + \delta x\, f(x_i + \tfrac{1}{2}\delta x, y_i + \tfrac{1}{2}\delta y) \\ x_{i+1} &= x_i + \delta x.\end{aligned}$$