Analytic Solution of the Finite Well Problem

We wish to solve the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

for a potential given by

$$V(x) = \begin{cases} 0 & (|x| < a)\,, \\ V_0 & (|x| > a)\,, \end{cases}$$

and $0 < E < V_0$ (i.e. bound states).

We start by recasting the equation in dimensionless form, by defining

$$s = x/a, \qquad \xi^2 = 2mEa^2/\hbar^2, \qquad U_0 = 2mV_0a^2/\hbar^2\,,$$

to obtain

$$-\frac{d^2\psi}{ds^2} = (\xi^2 - U)\,\psi\,,$$

where

$$U(s) = \begin{cases} 0 & (|s| < 1)\,, \\ U_0 & (|s| > 1)\,. \end{cases}$$

We will search for even and odd solutions separately. For $|s| < 1$, the even solution is

$$\psi(s) = A\cos\xi s\,.$$

The odd solution is

$$\psi(s) = B\sin\xi s\,.$$

For $|s| > 1$, the bounded solution is

$$\psi(s) = Ce^{-\eta s}\,,$$

where $\eta^2 = U_0 - \xi^2 = 2m(V_0 - E)a^2/\hbar^2$.

The condition at $s = 1$ is that ψ and ψ' should be continuous. Scaling out the overall normalization, this implies that ψ'/ψ is continuous. For the exterior solution, we have $\psi'/\psi = -\eta$, so the interior solution must satisfy

$$\psi' + \eta\,\psi = 0\,,$$

leading to the algebraic equations

$$-\xi\sin\xi + \eta\cos\xi = 0 \qquad (\psi \text{ even})$$

and

$$\xi\cos\xi + \eta\sin\xi = 0 \qquad (\psi \text{ odd})\,.$$