15

Maxwell's Equations

Contents

The electromagnetic field $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$ is determined by Maxwell's equations. These equations are linear in the space and time derivatives. In the momentum representation, obtained by taking a Fourier transform of the electric and magnetic fields, Maxwell's equations impose a set of four linear constraints on the six amplitudes $\mathbf{E}(k), \mathbf{B}(k)$. Why? At a more fundamental level, the electromagnetic field is described by photons. For each photon momentum state there are only two degrees of freedom, the helicity (polarization) states, corresponding to an angular momentum 1 aligned either in or opposite to the direction of propagation. Thus, the classical description of the electromagnetic field is profligate, introducing six amplitudes for each k when in fact only two are independent. The remaining four degrees must be absent in any description of a physically allowed field. The equations that annihilate these four nonphysical linear combinations are the equations of Maxwell. We derive these equations, in the absence of sources, by comparing the transformation properties of the helicity and classical field states for each four-momentum.

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Table 15.1.

Time Period	Approach	Strengths	Weaknesses
19th Century	Manifestly Covariant	Fields have elegant transformation properties	Many fields represent nonphysical states
20th Century	Hilbert Space	All linear superpositions represent physical states	Transformation properties are complicated

15.1 Introduction

The electromagnetic field has been described in two different ways. Following the nineteenth century approach (pre quantum mechanics), a field is introduced having appropriate transformation properties. The price one pays is that not every field represents a physically allowed state: such fields must be annihilated by appropriate equations. Following the twentieth century approach, a Hilbert space is introduced. An arbitrary superposition of states in this space represents a physically allowed field. The price one pays is that the field so constructed does not have obvious transformation properties.

In the older approach a field is defined at every point in space time. It is required to be "manifestly covariant." That is, it transforms as a tensor under homogeneous Lorentz transformations. This requires there to be a certain number of field components at every space-time point, or more conveniently, for every allowed momentum vector. In the Hilbert space formulation the number of independent components is just the allowed number of spin or helicity states. The number of components is never greater than the number of components required to define the "manifestly covariant" field; however, it may be less than this number. In this case there are linear combinations of the components of the manifestly covariant field that cannot represent physically allowed states. These linear combinations must be suppressed. It is the function of the field equations to suppress those linear combinations of components that do not correspond to physical states. These two approaches are compared in Table 15.1.

Maxwell's equations fulfill this function. The classical description involves six field components for each allowed mementum state. These are the classical electric and magnetic fields, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, or their components after Fourier transformation, $\mathbf{E}(k)$ and $\mathbf{B}(k)$, where k is a 4-vector that obeys $k\cdot k=\mathbf{k}\cdot\mathbf{k}-k_4k_4=0$. Here $\mathbf{k}$ is essentially a 3-momentum vector and k_4 is essentially an energy. The quantum description involves arbitrary superpositions of two helicity components for each momentum vector. The helicity states involve an angular momentum aligned along the direction of motion (helicity = +1 and right-handed polarization) and opposite the direction of propagation (helicity = −1 and left-handed polarization). There are four (= 6−2) linear combinations of classical field components that must be suppressed for each k-vector, and that are annihilated by Maxwell's equations. We derive these equations by comparing the transformation properties of the basis vectors for the 'manifestly covariant' but nonunitary representations of the inhomogeneous Lorentz group with the basis vectors for its unitary irreducible representations, which are not manifestly covariant. The set of constraints so derived reduce, for $j = 1$, to Maxwell's equations. This derivation is carried out for free fields (no sources) only. When sources are present the photon 4-vector k no longer obeys $k \cdot k = 0$. In this case the manifestly covariant equations provide a beautiful prescription for describing the coupling to source terms.

15.2 Review of the Inhomogeneous Lorentz Group

15.2.1 Homogeneous Lorentz Group

The wavefront for a light signal expanding from a source at the origin of coordinates for observers S and S' obeys the equation

$$x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2 = 0 \tag{15.1}$$

This requires that the coordinates (x, y, z, ict) and $(x, y, z, ict)'$ for observers S and S' be related by a homogeneous Lorentz transformation

$$\begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} = \begin{bmatrix} & & & \\ & \Lambda & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix}' \tag{15.2}$$

The 4×4 matrix transformations Λ belong to the Lie group $O(3, 1)$. The infinitesimal generators of a group operation in $SO(3, 1)$ are

$$\Lambda \rightarrow I_4 + \epsilon \begin{bmatrix} 0 & +\theta_3 & -\theta_2 & ib_1 \\ -\theta_3 & 0 & +\theta_1 & ib_2 \\ +\theta_2 & -\theta_1 & 0 & ib_3 \\ -ib_1 & -ib_2 & -ib_3 & 0 \end{bmatrix} = I_4 + \epsilon\,(\theta \cdot \mathbf{J} + \mathbf{b} \cdot \mathbf{K}) \tag{15.3}$$

Homogeneous Lorentz transformations leave invariant inner products: $k \cdot a = \Lambda k \cdot \Lambda a$, where k and a are four-vectors and $\Lambda \in O(3, 1)$. The infinitesimal generators $\mathbf{J}, \mathbf{K}$ satisfy the following commutation relations:

$$\begin{aligned} [J_i, J_j] &= -\epsilon_{ijk} J_k \\ [J_i, K_j] &= -\epsilon_{ijk} K_k \\ [K_i, K_j] &= +\epsilon_{ijk} J_k \end{aligned} \tag{15.4}$$

15.2.2 Inhomogeneous Lorentz Group

Intervals are preserved by the inhomogeneous Lorentz group:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (ct_2 - ct_1)^2 = \text{invariant} \tag{15.5}$$

The inhomogeneous Lorentz group consists of homogeneous Lorentz transformations, Λ, together with displacements of the origin. The general group transformation can be written as a 5×5 matrix, in terms of the 4-vector $a = (x, y, z, ct)$:

$$\{\Lambda, a\} = \left[\begin{array}{cccc|c} & & & & x \\ & \Lambda & & & y \\ & & & & z \\ & & & & ct \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{15.6}$$

as shown. The group composition law is matrix multiplication. The following results are immediate:

$$\begin{gathered} \{\Lambda_2, a_2\}\{\Lambda_1, a_1\} = \{\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1\} \\ \{I, a\}\{\Lambda, 0\} = \{\Lambda, a\} = \{\Lambda, 0\}\{I, \Lambda^{-1}a\} \end{gathered} \tag{15.7}$$

The inhomogeneous Lorentz group is the semidirect product of the homogeneous Lorentz group and the commutative invariant subgroup of translations of the origin of coordinates in space and time. The infinitesimal generators for this invariant subgroup are $(\partial/\partial x, \partial/\partial y, \partial/\partial z, i\partial/\partial(ct))$.

15.3 Subgroups and Their Representations

The group of inhomogeneous Lorentz transformations has two important subgroups. These are the subgroup of homogeneous Lorentz transformations $\{\Lambda, 0\}$ and the invariant subgroup of translations $\{I, a\}$. Both their representations play a role in the derivation of the relativistically covariant field equations.

15.3.1 Translations $\{I, a\}$

The translation subgroup $\{I, a\}$ is abelian (commutative). All of its unitary irreducible representations are one dimensional, and in fact

$$\Gamma^k(\{I, a\}) = e^{ik\cdot a} \tag{15.8}$$

where k is a 4-vector that parameterizes the one-dimensional representations. We may define a basis state for the one dimensional representation Γ^k of $\{I, a\}$ as $|k\rangle$:

$$\{I, a\}\,|k\rangle = |k'\rangle\langle k'|\,\{I, a\}\,|k\rangle = |k'\rangle\delta(k' - k)\Gamma^k(\{I, a\}) = |k\rangle e^{ik\cdot a} \tag{15.9}$$

Physically, k has a natural interpretation as the 4-momentum of the photon.

15.3.2 Homogeneous Lorentz Transformations

The Lie algebra $D_2 = A_1 + A_1$ is semisimple: it is the direct sum of two simple Lie algebras of type A_1 (c.f., Fig. 10.3). We can construct linear combinations of the infinitesimal generators $\mathbf{J}, \mathbf{K}$ of $SO(3,1)$ that are mutually commuting and that satisfy angular momentum commutation relations. These are

$$\begin{aligned} \mathbf{J}^{(1)} &= \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \\ \mathbf{J}^{(2)} &= \frac{1}{2}(\mathbf{J} + i\mathbf{K}) \end{aligned} \tag{15.10}$$

These operators satisfy angular momentum commutation relations

$$\begin{aligned} \left[\mathbf{J}_i^{(1)}, \mathbf{J}_j^{(1)}\right] &= -\epsilon_{ijk}\mathbf{J}_k^{(1)} \\ \left[\mathbf{J}_i^{(2)}, \mathbf{J}_j^{(2)}\right] &= -\epsilon_{ijk}\mathbf{J}_k^{(2)} \\ \left[\mathbf{J}_i^{(1)}, \mathbf{J}_j^{(2)}\right] &= 0 \end{aligned} \tag{15.11}$$

The algebra $\mathbf{J}^{(1)}$ has $2j+1$ dimensional irreducible representations D^j while $\mathbf{J}^{(2)}$ has $2j'+1$ dimensional irreducible representations $D^{j'}$. Any element in $SO(3,1)$ can be expressed in a $(2j+1)(2j'+1)$ dimensional representation $D^{jj'}$ as follows

$$\begin{aligned} &EXP(\theta\cdot\mathbf{J} + \mathbf{b}\cdot\mathbf{K}) = EXP\left[(\theta + i\mathbf{b})\cdot\mathbf{J}^{(1)} + (\theta - i\mathbf{b})\cdot\mathbf{J}^{(2)}\right] = \\ &D^j\left[(\theta + i\mathbf{b})\cdot\mathbf{J}^{(1)}\right] D^{j'}\left[(\theta - i\mathbf{b})\cdot\mathbf{J}^{(2)}\right] \end{aligned} \tag{15.12}$$

15.3.3 Representations of $SO(3,1)$

The Lie algebra $\mathfrak{so}(3,1)$ is isomorphic to the Lie algebra for the group of 2×2 matrices $SL(2;C)$. We have the following two isomorphisms

$$\begin{array}{rclcrcl} \mathbf{J} & = & \frac{i}{2}\sigma & \qquad & \mathbf{J} & = & \frac{i}{2}\sigma \\ \mathbf{K} & = & -\frac{1}{2}\sigma & & \mathbf{K} & = & +\frac{1}{2}\sigma \end{array} \tag{15.13}$$

These two isomorphisms give rise to the following two inequivalent sets of representations

$$\begin{array}{ccc} D^{j0} & \qquad & D^{0j} \\ \mathbf{K}^{(j)} = i\mathbf{J}^{(j)} & & \mathbf{K}^{(j)} = -i\mathbf{J}^{(j)} \end{array} \tag{15.14}$$

where $\mathbf{J}^{(j)}$ are the three $(2j+1)\times(2j+1)$ angular momentum matrices. The following matrices are associated with these representations

$$\begin{aligned} D^{j0}\left[\theta\cdot\mathbf{J}+\mathbf{b}\cdot\mathbf{K}\right] &= EXP\left[\theta\cdot\mathbf{J}^{(j)}+\mathbf{b}\cdot(+i\mathbf{J})^{(j)}\right] = EXP\left[(\theta+i\mathbf{b})\cdot\mathbf{J}^{(j)}\right] \\ D^{0j}\left[\theta\cdot\mathbf{J}+\mathbf{b}\cdot\mathbf{K}\right] &= EXP\left[\theta\cdot\mathbf{J}^{(j)}+\mathbf{b}\cdot(-i\mathbf{J})^{(j)}\right] = EXP\left[(\theta-i\mathbf{b})\cdot\mathbf{J}^{(j)}\right] \end{aligned} \tag{15.15}$$

These representations are complex conjugates of each other. The most general representation of $SO(3,1)$ is

$$D^{jj'}(\theta\cdot\mathbf{J}+\mathbf{b}\cdot\mathbf{K}) = EXP\left[(\theta+i\mathbf{b})\cdot\mathbf{J}^{(j)}\right]EXP\left[(\theta-i\mathbf{b})\cdot\mathbf{J}^{(j')}\right] = D^{jj'}(\Lambda) \tag{15.16}$$

Basis states for the action of Λ through the representation $D^{jj'}(\Lambda)$ can be computed

$$\Lambda\left|\begin{array}{cc} j & j' \\ \mu & \mu' \end{array}\right\rangle = \left|\begin{array}{cc} j & j' \\ \nu & \nu' \end{array}\right\rangle D^{jj'}_{\nu\nu';\mu\mu'}(\Lambda) \tag{15.17}$$

Under restriction to the subgroup $SO(3)\subset SO(3,1)$ this representation is reducible in a Clebsch-Gordan series

$$\begin{gathered} D^{jj'}(\Lambda) \stackrel{\Lambda\downarrow SO(3)}{\longrightarrow} D^{j}[SO(3)]\times D^{j'}[SO(3)] = \sum_{j''} D^{j''}[SO(3)] \\ |j-j'|\le j''\le j+j' \end{gathered} \tag{15.18}$$

This representation remains irreducible only if $j'=0$ or $j=0$.

15.4 Representations of the Poincaré Group

We construct here two kinds of representations for the inhomogeneous Lorentz group. These are the manifestly covariant representations and the unitary irreducible representations.

15.4.1 Manifestly Covariant Representations

A field $T_{\mu\nu}(x)$ is said to be **manifestly covariant** (obviously covariant) under transformations of the homogeneous Lorentz group $\Lambda \in SO(3,1)$ if

$$\Lambda T_{\mu\nu}(x) = T_{\mu'\nu'}(x\Lambda^{-1})\Lambda_{\mu'\mu}\Lambda_{\nu'\nu} \tag{15.19}$$

That is, the field components obviously form a basis on which the Lorentz transformation acts. The point at which the transformation acts is fixed, but since the coordinate system changes, the coordinates of the fixed point are changed by $x' = x\Lambda^{-1}$.

We construct manifestly covariant representations of the inhomogeneous Lorentz group by constructing direct products of basis vectors

$$|k\rangle \times | \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle \tag{15.20}$$

for the subgroups $\{I, a\}$ and $\{\Lambda, 0\}$ of the inhomogeneous Lorentz group. We define the action of the inhomogeneous Lorentz group on these direct product states by defining the action of the two subgroups, of homogeneous Lorentz transformations and of translations, on the momentum states $|k\rangle$ and the field component states $| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle$ separately.

We define the action of $\{I, a\}$ on these states by

$$\begin{array}{rcl} \{I,a\}\,|k\rangle & = & |k\rangle e^{ik\cdot a} \\ \{I,a\}\,| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle & = & | \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle \end{array} \tag{15.21}$$

The action of $\{\Lambda, 0\}$ on the momentum states follows from

$$\begin{array}{rcl} \{I,a\}\,[\{\Lambda,0\}\,|k\rangle] & = & \{\Lambda,0\}\left\{I,\Lambda^{-1}a\right\}|k\rangle \\ & = & [\{\Lambda,0\}\,|k\rangle]\, e^{ik\cdot\Lambda^{-1}a} \\ & = & [\{\Lambda,0\}\,|k\rangle]\, e^{i\Lambda k\cdot a} = |\Lambda k\rangle e^{i\Lambda k\cdot a} \end{array} \tag{15.22}$$

The action of $\{\Lambda, 0\}$ on the field component states is

$$\{\Lambda,0\}\,|\begin{array}{cc} j & j' \\ \mu & \mu' \end{array}\rangle = |\begin{array}{cc} j & j' \\ \nu & \nu' \end{array}\rangle D^{jj'}_{\nu\nu';\mu\mu'}(\Lambda) \tag{15.23}$$

If the vector space that carries a manifestly covariant representation of the inhomogeneous Lorentz group has the states

$$|k\rangle|\begin{array}{cc} j & j' \\ \mu & \mu' \end{array}\rangle \tag{15.24}$$

then all states of the form

$$|\Lambda k\rangle|\begin{array}{cc} j & j' \\ \nu & \nu' \end{array}\rangle \tag{15.25}$$

are also present in the underlying vector space.

The action of the two subgroups on the two types of states is summarized by

$$\begin{array}{c|ll} & |k\rangle & |\begin{array}{cc} j & j' \\ \mu & \mu' \end{array}\rangle \\ \hline \{I,a\} & |k\rangle e^{ik\cdot a} & |\begin{array}{cc} j & j' \\ \nu & \nu' \end{array}\rangle \delta_{\nu\nu';\mu\mu'} \\ \{\Lambda,0\} & |\Lambda k\rangle & |\begin{array}{cc} j & j' \\ \nu & \nu' \end{array}\rangle D^{jj'}_{\nu\nu';\mu\mu'}(\Lambda) \end{array} \tag{15.26}$$

15.4.2 Unitary Irreducible Representations

Suppose we have a representation of $\{\Lambda, a\}$ that is unitary and irreducible. Under restriction to the subgroup $\{I, a\}$ this reduces to a direct sum of irreducibles $\Gamma^k(\{I, a\})$ of $\{I, a\}$. The basis states are $|k;\xi\rangle$, where k is defined by the action of the translation $\{I, a\}$

$$\{I,a\}\,|k;\xi\rangle = |k;\xi\rangle e^{ik\cdot a} \tag{15.27}$$

and ξ is a helicity index that distinguishes different states with the same 4-momentum. A homogeneous Lorentz transformation maps the state $|k;\xi\rangle$ into a subspace of states parameterized by $k' = \Lambda k$

$$\begin{aligned} \{I,a\}\,\{\Lambda,0\}\,|k;\xi\rangle &= \{\Lambda,0\}\left\{I,\Lambda^{-1}a\right\}|k;\xi\rangle \\ &= \{\Lambda,0\}\,|k;\xi\rangle e^{ik\cdot\Lambda^{-1}a} \\ &= [\{\Lambda,0\}\,|k;\xi\rangle]\, e^{i\Lambda k\cdot a} \end{aligned} \tag{15.28}$$

As a result

$$\{\Lambda,0\}\,|k;\xi\rangle = |\Lambda k;\xi'\rangle M_{\xi'\xi}(\Lambda) \tag{15.29}$$

where $M_{\xi'\xi}(\Lambda)$ is a matrix that remains to be determined.

This simple calculation shows that if the 4-vector k parameterizes a state in an irreducible representation of the inhomogeneous Lorentz group, then the states k' with

$$k' = \Lambda k \tag{15.30}$$

are present also. To construct the matrix $M(\Lambda)$, we choose one particular 4-vector k^0 for each of the possible cases

$$\begin{array}{lccccccll} (i) & k\cdot k & > & 0 & & k^0 & = & (0,0,1,0) & \\ (ii) & k\cdot k & = & 0 & k\neq 0 & k^0 & = & (0,0,1,+i) & (a) \\ & & & & & k^0 & = & (0,0,1,-i) & (b) \\ (iii) & k\cdot k & < & 0 & & k^0 & = & (0,0,0,+i) & (a) \\ & & & & & k^0 & = & (0,0,0,-i) & (b) \\ (iv) & k\cdot k & = & 0 & k=0 & k^0 & = & (0,0,0,0) & \end{array} \tag{15.31}$$

The states (a), (b) are related to each other by the discrete time reversal operator T. The vector k^0 is called the **little vector**.

The effect of a homogeneous Lorentz transformation on the state $|k^0;\xi\rangle$ is determined by writing each Λ as a product of two group operations

$$\Lambda = C_k H_{k^0} \tag{15.32}$$

where

$$\begin{array}{rcl} H_{k^0}k^0 & = & k^0 \\ C_k k^0 & = & k \end{array} \tag{15.33}$$

That is, H_{k^0} is the stability subgroup of the little vector k^0 and C_k is a coset representative that maps k^0 into k:

$$C_k k^0 = k = \Lambda k^0 \tag{15.34}$$

The little groups (stability groups) of the little vectors k^0 are

(i) $SO(2,1)$
(ii) $ISO(2)$
(iii) $SO(3)$
(iv) $SO(3,1)$

These are determined as follows.

Case (i): An arbitrary element in the Lie subgroup acting on k^0 must leave k^0 invariant. Linearizing, an element in the Lie algebra must annihilate k^0:

$$\begin{bmatrix} 0 & +\theta_3 & -\theta_2 & ib_1 \\ -\theta_3 & 0 & +\theta_1 & ib_2 \\ +\theta_2 & -\theta_1 & 0 & ib_3 \\ -ib_1 & -ib_2 & -ib_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\theta_2 \\ +\theta_1 \\ 0 \\ -ib_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15.35}$$

The subalgebra leaving k^0 fixed is defined by $\theta_1 = \theta_2 = b_3 = 0$, θ_3, b_1, b_2 arbitrary. This is the three-dimensional subgroup $SO(2,1)$ consisting of generators for rotations about the z-axis and boosts in the x- and y-directions.

Case (ii): Applying the same arguments, we find

$$\begin{bmatrix} 0 & +\theta_3 & -\theta_2 & ib_1 \\ -\theta_3 & 0 & +\theta_1 & ib_2 \\ +\theta_2 & -\theta_1 & 0 & ib_3 \\ -ib_1 & -ib_2 & -ib_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} -\theta_2 - b_1 \\ +\theta_1 - b_2 \\ -b_3 \\ -ib_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15.36}$$

The stability subalgebra is defined by

$$\begin{array}{rcl} b_3 & = & 0 \\ b_2 & = & +\theta_1 \\ b_1 & = & -\theta_2 \end{array} \tag{15.37}$$

A general element in this subalgebra is

$$\begin{bmatrix} 0 & +\theta_3 & -\theta_2 & -i\theta_2 \\ -\theta_3 & 0 & +\theta_1 & i\theta_1 \\ +\theta_2 & -\theta_1 & 0 & 0 \\ i\theta_2 & -i\theta_1 & 0 & 0 \end{bmatrix} = \sum_i \theta_i Y_i \qquad \begin{array}{rcl} Y_1 & = & J_1 + K_2 \\ Y_2 & = & J_2 - K_1 \\ Y_3 & = & J_3 \end{array} \tag{15.38}$$

The operators Y_i obey the commutation relations

$$\begin{array}{rcl} [Y_3, Y_1] & = & -Y_2 \\ [Y_3, Y_2] & = & +Y_1 \\ [Y_1, Y_2] & = & 0 \end{array} \qquad ISO(2) \tag{15.39}$$

These are the commutation relations for the group $ISO(2)$, the group of inhomogeneous motions of the Euclidean plane R^2. Acting on the time-reversed little vector $(0,0,1,-i) = T(0,0,1,+i)$ the infinitesimal generators are $Y_1 = J_1 - K_2,\ Y_2 = J_2 + K_1,\ Y_3 = J_3$.

Case (iii): Proceeding as above

$$\begin{bmatrix} 0 & +\theta_3 & -\theta_2 & ib_1 \\ -\theta_3 & 0 & +\theta_1 & ib_2 \\ +\theta_2 & -\theta_1 & 0 & ib_3 \\ -ib_1 & -ib_2 & -ib_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \end{bmatrix} = \begin{bmatrix} -b_1 \\ -b_2 \\ -b_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15.40}$$

The subalgebra defined by $\mathbf{b} = 0$ is spanned by the angular momentum operators $\mathbf{J}$. It is $\mathfrak{su}(2)$.

Case (iv): This is the simplest case:

$$\begin{bmatrix} 0 & +\theta_3 & -\theta_2 & ib_1 \\ -\theta_3 & 0 & +\theta_1 & ib_2 \\ +\theta_2 & -\theta_1 & 0 & ib_3 \\ -ib_1 & -ib_2 & -ib_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15.41}$$

The little group of this vector is the entire homogeneous Lorentz group $SO(3,1)$.

The action of the little group on the subspace of states $|k^0;\xi\rangle$ is

$$\begin{aligned} H_{k^0}|k^0;\xi\rangle &= |H_{k^0}k^0;\xi'\rangle D_{\xi'\xi}(H_{k^0}) \\ &= |k^0;\xi'\rangle D_{\xi'\xi}(H_{k^0}) \end{aligned} \tag{15.42}$$

The original representation of the inhomogeneous Lorentz group is unitary and irreducible if and only if the representation $D_{\xi'\xi}(H_{k^0})$ of the little group is unitary and irreducible.

The cases (i) - (iv) are discussed here.

Case (i): The unitary irreducible representations of the noncompact group $SO(2,1)$ were described in problem 5 of Chapter 11. Since $k \cdot k > 0$ describes negative mass particles, we will not need to discuss these representations here.

Case (ii): See below.

Case (iii): The unitary irreducible representations for the group $SU(2)$, which is the little group for a massive particle at rest, were described in problem 2 of Chapter 6. They are described by an integer or half integer: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. The angular momentum j is a property of each massive particle.

Case (iv): The unitary irreducible representations of $SO(3,1)$ are known but not interesting for the present discussion.

We consider the case of zero mass particles in more detail here. The unitary irreducible representations of $ISO(2)$ are constructed following the prescription we are using to study the unitary irreducible representations of the inhomogeneous Lorentz group — the method of the little group. Since $ISO(2)$ has a two-dimensional translation invariant subgroup, basis states in a unitary irreducible representation can be labelled by a vector $\kappa = (\kappa_1, \kappa_2)$ in a two-dimensional Euclidean space, $\kappa \in R^2$, $\kappa \cdot \kappa \geq 0$. If a state $|\kappa\rangle$ is in one such representation, so are all states $|\kappa'\rangle$ for which $\kappa' \cdot \kappa' = \kappa \cdot \kappa$. That is, $\kappa' = (\kappa_1', \kappa_2')$ is related to $\kappa = (\kappa_1, \kappa_2)$ by a rotation: $\kappa' = R(\theta)\kappa$. The invariant length $\kappa \cdot \kappa$ parameterizes the representation. As before, two cases occur (c.f., Cases **(i)** or **(iii)** and Case **(iv)** above):

$$\begin{array}{rcclcl} (i) & \kappa \cdot \kappa & > & 0 & \text{little group} & = & \text{Identity} \\ (ii) & \kappa \cdot \kappa & = & 0 & \text{little group} & = & ISO(2) \end{array} \tag{15.43}$$

The first case presents us with two problems. First, κ^2 is a continuous quantum number, and there are no known particles with a continuous spin index. Second, if $\kappa^2 > 0$ there must be an infinite number of states with this same continuous index, for each 4-momentum value. Therefore we require $\kappa = 0$. This leaves us with the following physically allowable representations of the little group $(Y_1 \rightarrow 0, Y_2 \rightarrow 0)$

$$EXP(\theta_3 Y_3 + \theta_1 Y_1 + \theta_2 Y_2) = e^{i\xi\theta_3} \tag{15.44}$$

where ξ is an integer or half-integer.

The coset representatives C_k permute the 4-vector subspaces:

$$C_k|k^0;\xi\rangle = |k;\xi\rangle \tag{15.45}$$

The action of an arbitrary element of the inhomogeneous Lorentz group on any state in this Hilbert space is

$$\begin{array}{rcl} \{\Lambda, a\}\,|k;\xi\rangle & = & \{\Lambda, 0\}\left\{I, \Lambda^{-1}a\right\}|k;\xi\rangle \\ & = & \{\Lambda, 0\}\,|k;\xi\rangle e^{ik\cdot\Lambda^{-1}a} \\ & = & \{\Lambda, 0\}\,C_k|k^0;\xi\rangle e^{i\Lambda k\cdot a} \\ & = & \{\Lambda C_k, 0\}\,|k^0;\xi\rangle e^{i\Lambda k\cdot a} \\ & = & \{C_{k'}H_{k^0}, 0\}\,|k^0;\xi\rangle e^{i\Lambda k\cdot a} \\ & = & |k';\xi\rangle e^{i\xi\Theta}e^{i\Lambda k\cdot a} \end{array} \tag{15.46}$$

where

$$C_{k'}^{-1}\Lambda C_k = H_{k^0} = EXP(\Theta J_3 + \theta_1 Y_1 + \theta_2 Y_2) \longrightarrow e^{i\xi\Theta} \tag{15.47}$$

15.5 Transformation Properties

The Hilbert space that carries a unitary irreducible representation of a massless particle with helicity ξ contains all states of the form

$$|k;\xi\rangle \qquad \begin{array}{rcl} k & = & \Lambda k^0 \\ k^0 & = & (0,0,1,\pm i) \end{array} \tag{15.48}$$

The vector space that carries a manifestly covariant representation of a massless particle with transformation indices (j,j') contains all states of the form

$$|k\rangle| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle \qquad \begin{array}{rcl} k & = & \Lambda k^0 \\ k^0 & = & (0,0,1,\pm i) \end{array} \tag{15.49}$$

To compare these two ways of describing a massless particle we compare transformation properties of their states.

A. $\{H_{k^0},0\}$ **on** $|k^0;\xi\rangle$.

$$\{H_{k^0},0\}\,|k^0;\xi\rangle = |k^0;\xi\rangle e^{i\xi\Theta} \tag{15.50}$$

where $H_{k^0} = EXP(\Theta J_3 + \theta_1 Y_1 + \theta_2 Y_2)$.

B. $\{H_{k^0},0\}$ **on** $|k^0\rangle| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle$. The little group maps k^0 to k^0 but acts in a nontrivial way on the spin states

$$\{H_{k^0},0\}\,|k^0\rangle| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle = |k^0\rangle| \begin{array}{cc} j & j' \\ \nu & \nu' \end{array} \rangle D^{jj'}_{\nu\nu';\mu\mu'}(H_{k^0}) \tag{15.51}$$

The direct product representation $D^{jj'}$ has the following form

$$\begin{aligned} D^{j0}(H_{k^0}) &= EXP\left(\theta_3 J_3^{(j)} + \theta_1(J_1^{(j)} + iJ_2^{(j)}) + \theta_2(J_2^{(j)} - iJ_1^{(j)})\right) \\ &= EXP\left(\theta_3 J_3^{(j)} + (\theta_1 - i\theta_2)(J_1^{(j)} + iJ_2^{(j)})\right) \\ &= \left[\begin{array}{ccccc} e^{ij\theta_3} & \star & \star & \star & \star \\ & e^{i(j-1)\theta_3} & \star & \star & \star \\ & & \ddots & \star & \star \\ & & & \ddots & \star \\ & & & & e^{-ij\theta_3} \end{array}\right] \end{aligned} \tag{15.52}$$

$$
\begin{aligned}
D^{0j'}(H_{k^0}) &= EXP\left(\theta_3 J_3^{(j')} + \theta_1(J_1^{(j')} - iJ_2^{(j')}) + \theta_2(J_2^{(j')} + iJ_1^{(j')})\right) \\
&= EXP\left(\theta_3 J_3^{(j')} + (\theta_1 + i\theta_2)(J_1^{(j')} - iJ_2^{(j')})\right) \\
&= \begin{bmatrix} e^{ij'\theta_3} & & & & \\ \star & e^{i(j'-1)\theta_3} & & & \\ \star & \star & \ddots & & \\ \star & \star & \star & \ddots & \\ \star & \star & \star & \star & e^{-ij'\theta_3} \end{bmatrix}
\end{aligned} \qquad (15.53)
$$

By comparing Eq. (15.50) with Eq. (15.52) and Eq. (15.53) we reach the following conclusions:

The state $|k^0\rangle| \begin{matrix} j & 0 \\ j & 0 \end{matrix} \rangle$ transforms identically to $|k^0;\xi\rangle$ if $\xi > 0$ and $j = +\xi$.

The state $|k^0\rangle| \begin{matrix} 0 & j' \\ 0 & -j' \end{matrix} \rangle$ transforms identically to $|k^0;\xi\rangle$ if $\xi < 0$ and $j' = -\xi$.

If $|\psi\rangle$ is any physical state, it can be expanded in terms of either the helicity basis states $|k;\xi\rangle$ or the direct product states $|k\rangle| \begin{matrix} j & j' \\ \mu & \mu' \end{matrix} \rangle$:

$$
\begin{aligned}
|\psi\rangle &= \sum_{k,\xi} |k;\xi\rangle\langle k;\xi|\psi\rangle \\
|\psi\rangle &= \sum_{k,\mu\mu'} |k\rangle| \begin{matrix} j & j' \\ \mu & \mu' \end{matrix} \rangle\langle k; \begin{matrix} j & j' \\ \mu & \mu' \end{matrix} |\psi\rangle
\end{aligned}
$$

The amplitudes of the projection of $|\psi\rangle$ onto the basis states are $\langle k;\xi|\psi\rangle$ in the first case and $\langle k; \begin{matrix} j & j' \\ \mu & \mu' \end{matrix} |\psi\rangle$ in the second. In both cases the sum extends over all k vectors for which $\Lambda k \cdot \Lambda k = 0, k \neq 0$. In the first case the sum extends over the appropriate helicity states ξ ($\xi = \pm 1$ for photons). In the second case the sum extends over the appropriate values of $\mu, \mu' : \ -j \leq \mu \leq +j, \ -j' \leq \mu' \leq +j'$.

We discuss the positive helicity state $\xi = j > 0$ first. The amplitude $\langle k^0; j|\psi\rangle$ of the state $|k^0; j\rangle$ in any physical state $|\psi\rangle$ may be arbitrary. This is simply the amplitude of the massless particle of helicity j in the state $|\psi\rangle$. The amplitude $\langle k^0; \begin{matrix} j & 0 \\ j & 0 \end{matrix} |\psi\rangle$ in the same physical state $|\psi\rangle$ is the same. The amplitudes of the states $\langle k^0; \begin{matrix} j & 0 \\ m & 0 \end{matrix} |\psi\rangle$, $m \neq j$, must all vanish. These states are all superfluous — allowed in the manifestly covariant representation but not present in the Hilbert space that carries the unitary irreducible representation. A simple linear way to enforce this condition on the superfluous amplitudes is to require

$$\left\{ J_3^{(j)} k_3^0 - j k_4^0 I_{2j+1} \right\} \langle k^0; \begin{array}{cc} j & 0 \\ m & 0 \end{array} |\psi\rangle = 0 \qquad (15.54)$$

The matrix within the bracket $\{\cdot\}$ is diagonal, with the coefficient $(j - j)k_3^0 = 0$ multiplying the allowed amplitude $\langle k^0; \begin{array}{cc} j & 0 \\ j & 0 \end{array} |\psi\rangle$ and nonzero coefficients $(m-j)k_3^0$ multiplying the amplitudes $\langle k^0; \begin{array}{cc} j & 0 \\ m & 0 \end{array} |\psi\rangle$. Since $(m-j)k_3^0 \neq 0$, the amplitudes that are absent in the description of a physical state $(m \neq j)$ must vanish.

For the negative helicity states $\xi = -j$ we have by a completely similar argument

$$\left\{ J_3^{(j)} k_3^0 + j k_4^0 I_{2j+1} \right\} \langle k^0; \begin{array}{cc} 0 & j' \\ 0 & m' \end{array} |\psi\rangle = 0 \qquad (15.55)$$

C. Other k-vector subspaces.

The coset operator C_k maps the state $|k^0;\xi\rangle$ into the state

$$C_k |k^0;\xi\rangle = |k;\xi\rangle \qquad (15.56)$$

and the subspace $|k^0\rangle| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle$ into the subspace $|k\rangle| \begin{array}{cc} j & j' \\ \nu & \nu' \end{array} \rangle$ through the following nontrivial similarity transformation

$$C_k |k^0\rangle| \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} \rangle = |k\rangle| \begin{array}{cc} j & j' \\ \nu & \nu' \end{array} \rangle D^{jj'}_{\nu\nu';\mu\mu'}(C_k) \qquad (15.57)$$

The condition on the amplitude $\langle k; \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} |\psi\rangle$ in the subspace $|k\rangle$ is related to the conditions (15.54) and (15.55) in the subspace $|k^0\rangle$ by a similarity transformation

$$\begin{aligned} M^{jj'}(k^0)\langle k^0; \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} |\psi\rangle &= 0 \\ C_k M^{jj'}(k^0) C_k^{-1} \langle k; \begin{array}{cc} j & j' \\ \mu & \mu' \end{array} |\psi\rangle &= 0 \end{aligned} \qquad (15.58)$$

For the positive helicity state $\xi = j$ the matrix $M^{jj'}(k^0) = M^{j0}(k^0)$ is given in (15.54). The coset representative may be taken as the product of a boost in the z direction,

$$B_z(k)(0,0,1,i) = (0,0,k,ik) \tag{15.59}$$

followed by a rotation

$$R(k)(0,0,k,ik) = (k_1,k_2,k_3,ik_4) \qquad k_1^2+k_2^2+k_3^2 = k_4^2 = k^2 \tag{15.60}$$

For $j=1$ the similarity transformation becomes

$$R(\mathbf{k})B_z(k_4)\left\{J_3^{(j)} - jI_{2j+1}\right\}B_z^{-1}(k_4)R^{-1}(\mathbf{k}) =$$

$$\{\mathbf{J}\cdot\mathbf{k} - 1k_4I_3\}\,\langle k;\ \begin{matrix}1 & 0\\ \mu & 0\end{matrix}\ |\psi\rangle = 0 \tag{15.61}$$

as the linear constraint that must be satisfied in the subspace $|k\rangle|\ \begin{matrix}1 & 0\\ \mu & 0\end{matrix}\ \rangle$. The negative helicity states satisfy the constraint

$$\{\mathbf{J}\cdot\mathbf{k} + 1k_4I_3\}\,\langle k;\ \begin{matrix}0 & 1\\ 0 & \mu'\end{matrix}\ |\psi\rangle = 0 \tag{15.62}$$

15.6 Maxwell's Equations

The constraint equation is conveniently expressed in the coordinate rather than the momentum representation by inverting the original Fourier transform that brought us from the coordinate to the momentum representation

$$\langle k|x\rangle\left\{\mathbf{J}\cdot\frac{1}{i}\nabla + 1\frac{1}{i}\frac{\partial}{\partial(ict)}I_3\right\}\langle x|k\rangle\langle k;\ \begin{matrix}1 & 0\\ m & 0\end{matrix}\ |\psi\rangle = 0 \tag{15.63}$$

If we define complex fields $\langle x|k\rangle\langle k;\ \begin{matrix}j & 0\\ m & 0\end{matrix}\ |\psi\rangle$ by $\psi_{jm}(x)$, ($j=1$, $m = +1,0,-1$ or x,y,z or $1,2,3$) then this equation simplifies to a differential equation. In the standard representation for the angular momentum operators for $j=1$ we find

$$\begin{bmatrix} -\frac{i}{c}\frac{\partial}{\partial t} & +\partial_3 & -\partial_2 \\ -\partial_3 & -\frac{i}{c}\frac{\partial}{\partial t} & +\partial_1 \\ +\partial_2 & -\partial_1 & -\frac{i}{c}\frac{\partial}{\partial t} \end{bmatrix}\begin{bmatrix} B_1+iE_1 \\ B_2+iE_2 \\ B_3+iE_3 \end{bmatrix} = 0 \tag{15.64}$$

$$\begin{array}{rcrcrcl} -\frac{i}{c}\frac{\partial}{\partial t}(B+iE)_1 & + & \partial_3(B+iE)_2 & - & \partial_2(B+iE)_3 & = & 0 \\ -\partial_3(B+iE)_1 & - & \frac{i}{c}\frac{\partial}{\partial t}(B+iE)_2 & + & \partial_1(B+iE)_3 & = & 0 \\ +\partial_2(B+iE)_1 & - & \partial_1(B+iE)_2 & - & \frac{i}{c}\frac{\partial}{\partial t}(B+iE)_3 & = & 0 \end{array} \tag{15.65}$$

These three equations are summarized as a vector equation by

$$-\frac{i}{c}\frac{\partial}{\partial t}(\mathbf{B}+i\mathbf{E}) - \nabla\times(\mathbf{B}+i\mathbf{E}) = 0 \tag{15.66}$$

By taking the real and imaginary part of this complex equation we find

$$\begin{array}{lrcl} \mathrm{Re}: & +\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \nabla\times\mathbf{B} & = & 0 \\ \mathrm{Im}: & -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \nabla\times\mathbf{E} & = & 0 \end{array} \tag{15.67}$$

These are Maxwell's equations for positive helicity +1 massless particles (photons):

$$\begin{array}{rcl} \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} & = & 0 \\ \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} & = & 0 \end{array} \tag{15.68}$$

The equations for negative helicity states are derived from the complex conjugate representation D^{01} and are

$$\begin{bmatrix} +\frac{i}{c}\frac{\partial}{\partial t} & +\partial_3 & -\partial_2 \\ -\partial_3 & +\frac{i}{c}\frac{\partial}{\partial t} & +\partial_1 \\ +\partial_2 & -\partial_1 & +\frac{i}{c}\frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} B_1 - iE_1 \\ B_2 - iE_2 \\ B_3 - iE_3 \end{bmatrix} = 0 \tag{15.69}$$

It is easily verified that the resulting equations are identical to Eq. (15.68).

15.7 Conclusion

In some sense, Maxwell's equations were a historical accident. Had the discovery of Quantum Mechanics preceeded the unification of electricity and magnetism, Maxwell's equations might not have loomed so large in the history of Physics.

In the quantum description of the electromagnetic field, photons are the fundamental building blocks. Photons are described by a 4-vector k that obeys $k \cdot k = 0$ in free space, and a helicity index indicating a projection of an angular momentum ± 1 along the direction of propagation of the photon. Every physical state is described by a superposition of the photon basis states, and every superposition describes a possible physical state. In this description of the electromagnetic field in free space no constraint equations are necessary.

The 19th century description of the electromagnetic field proceeds along somewhat different lines. A multicomponent field $(\mathbf{E}, \mathbf{B})$ is introduced at each point in space-time. The components of the field transform in a very elegant way under homogeneous Lorentz transformations (as a tensor). If the field is Fourier transformed from the coordinate to the momentum representation, then each 4-momentum has six components associated with it. These are the components of a second order antisymmetric tensor. Since the quantum description has only two independent components associated with each 4-momentum, there are four dimensions worth of linear combinations of the classical field components that do not describe physically allowed states, for each 4-momentum. Some mechanism must be derived for annihilating these superpositions. This mechanism is the set of equations discovered by Maxwell. In this sense, Maxwell's equations are an expression of our ignorance.

It is ironic that the first truly powerful applications of group theory were to the solutions of equations. We now understand that group theory, by pointing to the appropriate Hilbert space for the electromagnetic field, allows us to relate physical states to arbitrary superpositions of basis states. Since no superpositions are forbidden, no equations are necessary.

15.8 Problems

1. So — where are the divergence equations? In the special frame with little vector $k^0 = (0,0,1,i)$ the only nonvanishing component of the field, $\langle k; \begin{matrix} j=1 & 0 \\ m & 0 \end{matrix} |\psi\rangle$, is the component with $m = +1$ (c.f., Eq. (15.54)). The coordinates are $-(v_x + iv_y)$. The vector $\mathbf{v} = (v_x, v_y, 0)$ represented by this coordinate is orthogonal to the spacial part of the little vector $\mathbf{k}^0 = (0,0,1)$: $\mathbf{k}^0 \cdot \mathbf{v} = 0$. Under boosts B_z and rotations, the nonvanishing component of the boosted field is orthogonal to the spacial part of the $\mathbf{k}$ vector: $\mathbf{k} \cdot \mathbf{v}(\mathbf{k}) = 0$. Backtransforming from the Fourier to the spacial representation please find that

$$\mathbf{k} \cdot \mathbf{v}(\mathbf{k}) = 0 \xrightarrow{\mathrm{FT}^{-1}} \nabla \cdot (\mathbf{B} + i\mathbf{E}) = 0$$

Taking the real and imaginary parts of this equation give the source-free divergence equations $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$. Show this.

2. When sources are present the Maxwell equations are modified in a way that is most clearly expressed in the "manifestly covariant representation." If particle j at $\mathbf{x}(j)$ has electric charge e_j and magnetic charge m_j, the electric and magnetic charge densities and current densities are defined by

	Electric	Magnetic
Charge Density	$\rho_e(\mathbf{x},t) = \sum_j e_j \mathbf{x}_j(t)$	$\rho_m(\mathbf{x},t) = \sum_j m_j \mathbf{x}_j(t)$
Current Density	$\mathbf{J}_e(\mathbf{x},t) = \sum_j e_j \frac{d\mathbf{x}_j(t)}{dt}$	$\mathbf{J}_m(\mathbf{x},t) = \sum_j m_j \frac{d\mathbf{x}_j(t)}{dt}$
Conservation Law	$\nabla \cdot \mathbf{J}_e(\mathbf{x},t) + \frac{\partial \rho_e(\mathbf{x},t)}{\partial t} = 0$	$\nabla \cdot \mathbf{J}_m(\mathbf{x},t) + \frac{\partial \rho_m(\mathbf{x},t)}{\partial t} = 0$

The conservation equations enforce the conditions of charge conservation (both electric and magnetic, separately).

In order to extend Maxwell's equations to include sources, the source free (homogeneous) equations (15.66) must be coupled to the source terms in such a way that the symmetry properties on the left (the fields) match the symmetry properties of the sources. Thus, the right-hand side must include only vector terms, and these terms must have appropriate transformation properties under the discrete operations T, P, TP. The result is unique up to scale factor:

$$(\nabla \times + \frac{i}{c}\frac{\partial}{\partial t})(\mathbf{B} + i\mathbf{E}) = \frac{1}{i}\,\frac{4\pi}{c}(\mathbf{J}_m + i\mathbf{J}_e) \tag{15.70}$$

The factor 4π is the surface area of the unit sphere in R^3, and the factor $1/c$ on the right is determined by the system of units used (Gaussian).

a. Show that Maxwell's equations with sources are

$$\begin{array}{lcccc} \nabla \times \mathbf{B} & - & \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} & = & +\frac{4\pi}{c}\mathbf{J}_e \\[2ex] \nabla \times \mathbf{E} & + & \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} & = & -\frac{4\pi}{c}\mathbf{J}_m \end{array}$$

b. Show that the Maxwell equations with sources are invariant under the simultaneous transformation

$$\begin{array}{rclcl} \mathbf{B}+i\mathbf{E} & \rightarrow & \mathbf{B}'+i\mathbf{E}' & = & e^{i\phi}(\mathbf{B}+i\mathbf{E}) \\ \mathbf{J}_m+i\mathbf{J}_e & \rightarrow & \mathbf{J}_m^{'}+i\mathbf{J}_e^{'} & = & e^{i\phi}(\mathbf{J}_m+i\mathbf{J}_e) \end{array}$$

In particular, show that for $\phi = \pi/2$ this is the dual transformation $(\mathbf{B},\mathbf{E}) \rightarrow (\mathbf{E},-\mathbf{B})$.

c. Take the divergence of both sides of Eq. (15.70). Use the vector identity div curl (*) =0, for * = anyvector. Show

$$\frac{i}{c}\frac{\partial}{\partial t}\left\{\nabla\cdot(\mathbf{B}+i\mathbf{E})-4\pi(\rho_m+i\rho_e)\right\}=0$$

d. By taking real and imaginary parts and integrating over time, find the following:

$$\begin{array}{rclcl} \nabla\cdot\mathbf{B}(\mathbf{x},t) & = & 4\pi\rho_m(\mathbf{x},t) & + & C_m(\mathbf{x}) \\ \nabla\cdot\mathbf{E}(\mathbf{x},t) & = & 4\pi\rho_e(\mathbf{x},t) & + & C_e(\mathbf{x}) \end{array}$$

e. Two "constants of integration" appear in these equations. They are functions of space but not of time. If these "constant functions of position" are zero the Maxwell divergence equations result. Provide arguments to show that these constants should be zero. These should take the form of investigating what the field looks like when all particles head towards "infinity."

Remark. So far magnetic charges (monopoles) have not been observed, despite their predictions by supersymmetric theories and active, difficult searches by experimentalists. This means that the first divergence equation is $\nabla\cdot\mathbf{B}=0$.

3. In order to describe gravitational waves in free space it is possible to use the representation $D^{jj'+j'j}(\Lambda)$, with $j-j'=\pm 2$. In the case with $(j,j')=(2,0)$ a curl equation is introduced to suppress four nonphysical complex amplitudes. Show that the gravitational wave equations in free space are

$$-\frac{2i}{c}\frac{\partial}{\partial t}(\mathbf{G_m}+i\mathbf{G_e})-\nabla\times(\mathbf{G_m}+i\mathbf{G_e})=0 \tag{15.71}$$

The real and imaginary part of this complex equation are

$$\begin{array}{lrcl} \mathrm{Re}: & +\dfrac{2}{c}\dfrac{\partial\mathbf{G_e}}{\partial t}-\nabla\times\mathbf{G_m} & = & 0 \\ \mathrm{Im}: & -\dfrac{2}{c}\dfrac{\partial\mathbf{G_m}}{\partial t}-\nabla\times\mathbf{G_e} & = & 0 \end{array} \tag{15.72}$$

The fields $\mathbf{G_e}$ and $\mathbf{G_m}$ are called the gravitoelectric and gravitomagnetic fields. These fields can be treated in Cartesian coordinates as real symmetric 3×3 traceless matrices and in spherical coordinates as 5-component rank-2 spherical tensors. In the latter case the curl operator is $\mathbf{J}\cdot\nabla$, where $\mathbf{J}$ is the 5×5 angular momentum operator:

$$\mathbf{J}\cdot\nabla = \begin{bmatrix} +2\partial_0 & \sqrt{4}\partial_+ & 0 & 0 & 0 \\ \sqrt{4}\partial_- & +1\partial_0 & \sqrt{6}\partial_+ & 0 & 0 \\ 0 & \sqrt{6}\partial_- & 0\partial_0 & \sqrt{6}\partial_+ & 0 \\ 0 & 0 & \sqrt{6}\partial_- & -1\partial_0 & \sqrt{4}\partial_+ \\ 0 & 0 & 0 & \sqrt{6}\partial_- & -2\partial_0 \end{bmatrix}$$

In Cartesian coordinates the curl operator is slightly more complicated. The Maxwell-like equations for the gravito- electric/magnetic field are

$$\begin{bmatrix} 0 & \partial_y & -\partial_x & 2\partial_z & 0 \\ -\partial_y & 0 & \partial_z & -\partial_x & -\sqrt{3}\partial_x \\ \partial_x & -\partial_z & 0 & -\partial_y & \sqrt{3}\partial_y \\ -2\partial_z & \partial_x & \partial_y & 0 & 0 \\ 0 & \sqrt{3}\partial_x & -\sqrt{3}\partial_y & 0 & 0 \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} + \frac{2}{c}\frac{\partial}{\partial t} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{pmatrix} = 0$$

$$\begin{bmatrix} 0 & \partial_y & -\partial_x & 2\partial_z & 0 \\ -\partial_y & 0 & \partial_z & -\partial_x & -\sqrt{3}\partial_x \\ \partial_x & -\partial_z & 0 & -\partial_y & \sqrt{3}\partial_y \\ -2\partial_z & \partial_x & \partial_y & 0 & 0 \\ 0 & \sqrt{3}\partial_x & -\sqrt{3}\partial_y & 0 & 0 \end{bmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{pmatrix} - \frac{2}{c}\frac{\partial}{\partial t} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = 0$$

The relation between the five components of the rank-2 spherical tensor and the nine matrix elements of a second order Cartesian tensor are [57]

$$F_{ij} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} F_4 - \frac{1}{\sqrt{3}}F_5 & F_1 & F_3 \\ F_1 & -F_4 - \frac{1}{\sqrt{3}}F_5 & F_2 \\ F_3 & F_2 & +\frac{2}{\sqrt{3}}F_5 \end{pmatrix}$$

The matrix components obey $F_{ij} = F_{ji}$, $\sum_i F_{ii} = 0$, and $\partial^i F_{ij} = 0$. The gravito-electric and magnetic tensors have the same discrete symmetries as the electric and magnetic fields.

4. Follow the outline of Problem #2 to show

a. The gravito-electric and magnetic fields satisfy divergence conditions in free space. Write them down.

b. In the presence of source terms (stationary and moving masses) the homogeneous equations are "dressed" with source terms on the right hand side. In Cartesian coordinates the source term for the gravito-electric field is $U_{ij} = \sum_k m_k(\mathbf{x}_k(t)\mathbf{x}_k(t))_{ij}$, and the form of the rank-2 tensor is determined from the expression at the conclusion of Problem #4. What is the gravitational analog of the magnetic monopole?

c. The coupled equations are invariant under a gauge transformation of the first kind of both the gravito-electric and magnetic fields and the current terms: $\mathbf{G_m} + i\mathbf{G_e} \rightarrow e^{i\phi}(\mathbf{G_m} + i\mathbf{G_e})$ and $\mathbf{J_m} + i\mathbf{J_e} \rightarrow e^{i\phi}(\mathbf{J_m} + i\mathbf{J_e})$. Show this.

d. What are the divergence equations in the presence of moving matter?

5. Construct the source-free field equations for gravitons for the $D^{jj}(\Lambda)$ representation, with $j = 1$. Show that there are seven constraints that correspond to (J, M) with $(J, M) = (0, 0), (1, 0), (1, \pm 1), (2, 0), (2 \pm 1)$. What are these equations in the standard differential representation? How are source terms (moving masses) coupled to these equations?

6. Observed redshifts are extremely important in interpreting the history of our universe. There appear to be four sources for redshifts (so far):

a. Döppler shift;

b. Gravitational redshift;

c. Universal Expansion redshift;

d. Mach redshift.

The Döppler shift has been recognized since 1842. Radiation from a source is redshifted if the source and observer are moving away form each other, blueshifted if they are moving towards each other. The gravitational redshift is a consequence of the conservation of energy. As a photon climbs out of a gravitational potential it loses energy and its frequency is redshifted. The universal expansion redshift is a consequence of the expansion of the universe. Two points (e.g., a source and an observer) that are at rest with respect to the the COBE background radiation (the "aether") move apart due to the expansion of the universe. If a wave with N wavelengths connects the two (distance $N\lambda$), as time goes on and the distance increases the wavelength must also increase to $N\lambda'$. This redshift source is sometimes confused with the Döppler shift because the two points appear to be moving apart due to the expansion of the universe. The fourth redshift source is controversial. Mach proposed that the inertia (mass) of a particle depends on the distribution of mass in the universe. Field theory requires that this information is transmitted by the fields set up by charges (electric, magnetic (if they exist), and masses). In fact, the exchange of virtual gravitons provides information about the distribution of mass in the universe within our horizon and should contribute to the mass (inertia) of a particle in the same way that exchange of virtual photons contributes to the energy (mass) changes in the Lamb effect.

a. Assume that the energy density in the universe has the form $\rho(\mathbf{x}, t) = \rho(t)$ (time dependent only). Assume that since recombination ($\sim$ 300KY after the Big Bang) the horizon of the accessible universe has been uniformly expanding. Assume that the mass of the electron comes from two sources: interactions with electromagnetic radiation and interaction with graviational radiation. Compute how the mass changes with time.

b. Estimate the mass-dependence of the electron-proton mass ratio $m_e(t)/M_p(t)$.

c. If the electron mass is increasing in time because of the expansion of the horizon with time, then the electron was less massive in the past. Radiation emitted from the hydrogen atom has frequency $\nu = \frac{1}{2}(mc^2/\hbar) \times |(1/n_1^2 - 1/n_2^2)|$ where n_1 and n_2 are the principle quantum numbers of the two states involved in the transition and m is the reduced mass of the electron-proton system. Show that H_α photons emitted from hydrogen at rest with the COBE background are redshifted because of the Universal Expansion *and* because the electron was less massive in the past. Disentangle these two effects and argue that the Mach shift aliases the Universal Expansion redshift.

7. The locally flat metric of space time and the metric representing a certain type of gravitational field are given by the matrices

$$g_{\text{flat}} = \left[\begin{array}{c|ccc} c^2 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] \qquad g_{\text{grav.}} = \left[\begin{array}{c|ccc} c^2\left(1 + \dfrac{2\Phi(x)}{c^2}\right) & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right]$$

Here $\Phi(x)$ is the local Newtonian gravitational field. Find a locally linear coordinate transformation S that brings the curved metric to flat form: $S^t g_{\text{grav.}} S = g_{\text{flat}}$. Interpret S in terms of a locally free-falling coordinate transformation.

8. **Gauss' Law on the Sphere S^2.** Gauss' Law in R^3 states

$$\oint \mathbf{E}\cdot d\mathbf{S} = \int 4\pi\rho \, dV$$

The integral on the left is over the surface bounding the volume V over which the integral on the right extends, $\mathbf{E}$ is the electric field and ρ is the charge density. For a charge q at the origin of a sphere of radius a, $\rho(x) = q\delta(x)$, The $\mathbf{E}$ field is spherically symmetric, and Gauss' Law reduces to

$$4\pi a^2 |\mathbf{E}(a)| = 4\pi q$$

From this, and symmetry, we deduce the Coulomb/gravitational force law:

$$\mathbf{E}(a) = \frac{q}{a^2}\frac{\mathbf{a}}{|\mathbf{a}|}$$

By completely similar arguments Gauss' Law in the plane R^2 gives $|\mathbf{E}(a)| = q/|\mathbf{a}|$.

Assume a Gauss Law ($\oint \mathbf{E}\cdot d\mathbf{S} = \int 2\pi\rho dA$) holds on the sphere S^2. Place a charge q on the north pole of a sphere of radius R (c.f., Fig. 15.1).

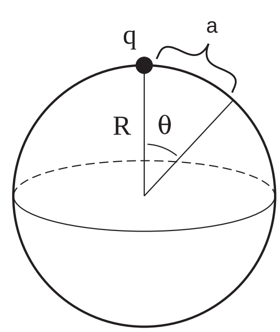

Fig. 15.1. A charge q is placed on the north pole of a sphere of radius R.

a. An observation point subtends an angle θ when measured from the center of the sphere S^2 (c.f., Fig. 15.1). Show that its distance a from the north pole is $a = R\theta$ and the circumference of a circle of latitude through this point is $2\pi R \sin\theta$. Use this information to deduce

$$|\mathbf{E}| = \frac{q}{R\sin\theta} = \frac{q}{R\sin(a/R)}$$

Conclude that the field is stronger than the q/a form it would have in a plane.

b. Show that this effective strengthening is due to the relative compression of the **E** field lines (compared to the planar case) due to the positive curvature of the sphere.

c. Rewrite this result as

$$|\mathbf{E}| = \frac{q}{R\sin(a/R)} = \frac{q(a)}{a} \qquad q(a) = q\left(\frac{a/R}{\sin(a/R)}\right)$$

where a $(= R\theta)$ is the distance from the charge to the observation point.

d. If the observer thinks (s)he is in a flat space, conclude (s)he will think the effective charge depends on the distance from the observation point. In particular, if $a = ct$, the further back in time the observer looks, the stronger (s)he will think the charge is.

9. **Gauss' Law on Rank-One Homogeneous Spaces:** The invariant metric and measure on the three Riemannian symmetric spaces $H^n = SO(n,1)/\ SO(n)$, $R^n = ISO(n)/SO(n)$, and $S^n = SO(n+1)/SO(n)$ are

$$ds^2 = \frac{dr^2}{1-kr^2} + r^2\sum_{j=2}^{n}(\sin\theta_2\sin\theta_3\ldots\sin\theta_{j-1}d\theta_j)^2$$

where $k = (-1, 0, +1)$ for H^n, R^n, S^n and radial coordinates are used:

$$\begin{aligned}
x_1 &= r\cos\theta_2 \\
x_2 &= r\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_{n-1} &= r\sin\theta_2\sin\theta_3\ldots\sin\theta_{n-1}\cos\theta_n \\
x_n &= r\sin\theta_2\sin\theta_3\ldots\sin\theta_{n-1}\sin\theta_n
\end{aligned}$$

a. Derive the metric for H^n, S^n from Eq. (12.9) and the coordinate transformation above.

b. Assume a Gauss Law of the form

$$\oint \mathbf{E}\cdot d\mathbf{S} = \int \Omega\rho(x)dV$$

Compute Ω, the surface area of the unit sphere $S^{n-1} \subset H^n, R^n$, or S^n. (Hint: use $\int e^{-x^2}dx = \sqrt{\pi}$, carry the n-fold integral out in Cartesian and radial coordinates, and show $\Omega = 2\pi^{n/2}/\Gamma(n/2)$.)

c. Carry out the integral for a charge q at the origin to show

$$|\mathbf{E}|a^{n-1} = q$$

d. Show that the distance d from the origin to the sphere of radius a is

$$d(a) = \int_0^a \frac{dr}{\sqrt{1-kr^2}} \longrightarrow \begin{array}{rcrcr} \mathrm{Sinh}^{-1}a & & k & = & -1 \\ a & & k & = & 0 \\ \mathrm{Sin}^{-1}a & & k & = & +1 \end{array}$$

e. Express the electric field strength as

$$|\mathbf{E}| = \frac{q(d)}{d^{n-1}} \qquad q(d) = q \times \begin{array}{c} \left(\frac{d/R}{\sinh(d/R)}\right)^{n-1} \\ 1 \\ \left(\frac{d/R}{\sin(d/R)}\right)^{n-1} \end{array}$$

Here R is some characteristic size scale for the spaces H^n, S^n.

f. Show that in the two curved spaces the observed charge is renormalized upward in S^n, downward in H^n, with lookback time. Give a physical interpretation involving compression or rarefaction of field lines. How does this renormalization depend on R, c, t?

10. The Special Theory of Relativity is based on two assumptions that have been raised to the status of axioms:

1. The speed of light is the same in all inertial frames.
2. Physical laws have the same form in all inertial frames.

The second axiom has been rephrased in the spirit of thermodynamics: "It is impossible, by any experiment, to determine the absolute motion of an inertial frame of reference." This form is motivated by the failure of the Michelson-Morley experiment to detect the motion of the earth through the "aether." In this form the second axiom is false: This has been shown by measurements of the microwave background radiation, which contains a nonzero dipole moment. This shows that the Local Group of galaxies is moving through the microwave background at a speed of $\sim$ 380 km/sec in the direction of the constellation Leo.

a. What effect does the ability to determine an absolute frame of reference have on the Special Theory of Relativity?

b. Assume the temperature distribution of the microwave background is $T(\theta, \phi; t) = \sum_{l,m} A^l_m(t) Y^l_m(\theta, \phi)$. How do you use this information to determine a frame that is: Not translating? Not rotating?

c. Since an absolute rest frame (non translating, non rotating) is defined by thermodynamic measurements, argue that this special reference frame is statistically determined.

d. Show that the determination of this special frame of reference is uncertain due to the Uncertainty Relations of Statistical Mechanics: $\Delta U \; \Delta(1/T) \geq k$ in the entropy representation [33].

e. If thermodynamic background fields of spin $\frac{1}{2}$ (neutrinoes) and spin 2 (gravitons) also exist, show that they also can be used to determine special rest frames. Argue why, or why not, the special frames defined by $j = \frac{1}{2}, 1, 2$ are the same. What happens if they are different?

f. Assume (for simplicity) that there is only one massive object in the universe and that it moves through the microwave background radiation with a velocity $v(t)$. Show that its velocity decays to zero according to $v(t) \simeq v(t_0) e^{-(t-t_0)/\tau}$ because it is moving through a viscous medium. Estimate τ and present your answer in the form τ/T_p, where T_p is the present age of the universe ($T_p \simeq 13.7$ BY). To carry out this estimate you may assume the massive object is a black body — in fact, assume it is a black hole with mass M, radius R at temperature T_{BH}. Use the standard relations for a neutral nonrotating black hole $R = 2GM/c^2$, $T_{BH} = \hbar c^3/8\pi kGM$. You can assume that the mass M is sufficiently large that the temperature T_{BH} can be neglected (set to zero). Assume that the absorption (geometric) cross section for radiation on a black hole is $\gamma \pi R^2$, where $\gamma = 3^3/2^2$. Note that the problem of slowing down in a viscous medium was discused by Einstein in another of the papers from his "annus mirabilis", the precursor of the Fluctuation-Dissipation Theorem.