4

Lie Algebras

Contents

The study of Lie groups can be greatly facilitated by linearizing the group in the neighborhood of its identity. This results in a structure called a Lie algebra. The Lie algebra retains most, but not quite all, of the properties of the original Lie group. Moreover, most of the Lie group properties can be recovered by the inverse of the linearization operation, carried out by the EXPonential mapping. Since the Lie algebra is a linear vector space, it can be studied using all the standard tools available for linear vector spaces. In particular, we can define convenient inner products and make standard choices of basis vectors. The properties of a Lie algebra in the neighborhood of the origin are identified with the properties of the original Lie group in the neighborhood of the identity. These structures, such as inner product and volume element, are extended over the entire group manifold using the group multiplication operation.

4.1 Why Bother?

Two Lie groups are isomorphic if:

(i) Their underlying manifolds are topologically equivalent;

(ii) The functions defining the group composition laws are equivalent.

Two manifolds are topologically equivalent if they can be smoothly deformed into each other. This requires that all their topological indices, such as dimension, Betti numbers, connectivity properties, etc., are equal.

Two group composition laws are equivalent if there is a smooth change of variables that deforms one function into the other.

Showing the topological equivalence of two manifolds is not necessarily an easy job. Showing the equivalence of two composition laws is typically a much more difficult task. It is difficult because the group composition law is generally nonlinear, and working with nonlinear functions is notoriously difficult.

The study of Lie groups would simplify greatly if the group composition law could somehow be linearized, and this linearization retained a substantial part of the information inherent in the original group composition law. This in fact can be done.

Lie algebras are constructed by linearizing Lie groups.

A Lie group can be linearized in the neighborhood of any of its points, or group operations. Linearization amounts to Taylor series expansion about the coordinates that define the group operation. What is being Taylor expanded is the group composition function. This function can be expanded in the neighborhoods of any group operations.

A Lie group is homogeneous — every point looks locally like every other point. This can be seen as follows. The neighborhood of group element a can be mapped into the neighborhood of group element b by multiplying a, and every element in its neighborhood, on the left by group element ba^{-1} (or on the right by $a^{-1}b$). This maps a into b and points near a into points near b.

It is therefore necessary to study the neighborhood of only one group operation in detail. Although geometrically all points are equivalent, algebraically one is special — the identity. It is very useful and convenient to study the neighborhood of this special group element.

Linearization of a Lie group about the identity generates a new set of operators. These operators form a **Lie algebra**. A Lie algebra is a linear vector space, by virtue of the linearization process.

The composition of two group operations in the neighborhood of the identity reduces to vector addition. The construction of more complicated group products, such as the commutator, and the linearization of these products introduces additional structure in this linear vector space. This additional structure, the commutation relations, carries information about the original group composition law.

In short, the linearization of a Lie group in the neighborhood of the identity to form a Lie algebra brings about an enormous simplification in the study of Lie groups.

4.2 How to Linearize a Lie Group

We illustrate how to construct a Lie algebra for a Lie group in this section. The construction is relatively straightforward once an explicit parameterization of the underlying manifold and an expression for the group composition law is available. In particular, for the matrix groups the group composition law is matrix multiplication, and one can construct the Lie algebra immediately for the matrix Lie groups.

We carry this construction out for $SL(2;R)$. It is both customary and convenient to parameterize a Lie group so that the origin of the coordinate system maps to the identity of the group. Accordingly, we parameterize $SL(2;R)$ as follows

$$(a,b,c) \longrightarrow M(a,b,c) = \begin{bmatrix} 1+a & b \\ c & (1+bc)/(1+a) \end{bmatrix} \tag{4.1}$$

The group is linearized by investigating the neighborhood of the identity. This is done by allowing the parameters (a,b,c) to become infinitesimals and expanding the group operation in terms of these infinitesimals to first order

$$(a,b,c) \rightarrow (\delta a,\delta b,\delta c) \rightarrow M(\delta a,\delta b,\delta c) = \begin{bmatrix} 1+\delta a & \delta b \\ \delta c & (1+\delta b\delta c)/(1+\delta a) \end{bmatrix} \tag{4.2}$$

The basis vectors in the Lie algebra are the coefficients of the first order infinitesimals. In the present case the basis vectors are 2×2 matrices

$$(\delta a,\delta b,\delta c) \rightarrow I_2 + \delta a X_a + \delta b X_b + \delta c X_c = \begin{bmatrix} 1+\delta a & \delta b \\ \delta c & 1-\delta a \end{bmatrix} \tag{4.3}$$

$$\begin{aligned} X_a &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left.\frac{\partial M(a,b,c)}{\partial a}\right|_{(a,b,c)=(0,0,0)} \\ X_b &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left.\frac{\partial M(a,b,c)}{\partial b}\right|_{(a,b,c)=(0,0,0)} \\ X_c &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \left.\frac{\partial M(a,b,c)}{\partial c}\right|_{(a,b,c)=(0,0,0)} \end{aligned} \tag{4.4}$$

Lie groups that are isomorphic have Lie algebras that are isomorphic.

Remark: The group composition function $\phi(x, y)$ is usually linearized in one of its arguments, say $\phi(x, y) \rightarrow \phi(x, 0 + \delta y)$. This generates a left-invariant vector field. The commutators of two left-invariant vector fields at a point x is independent of x, so that x can be taken in the neighborhood of the identity. It is for this reason that the linearization of the group in the neighborhood of the identity is so powerful.

4.3 Inversion of the Linearization Map: EXP

Linearization of a Lie group in the neighborhood of the identity to form a Lie algebra preserves the local group properties but destroys the global properties — that is, what happens far from the identity. It is important to know whether the linearization process can be reversed — can one recover the Lie group from its Lie algebra?

To answer this question, assume X is some operator in a Lie algebra — such as a linear combination of the three matrices spanning the Lie algebra of $SL(2; R)$ given in (4.4). Then if ϵ is a small real number, $I + \epsilon X$ represents an element in the Lie group close to the identity. We can attempt to move far from the identity by iterating this group operation many times

$$\lim_{k\to\infty}(I + \frac{1}{k}X)^k = \sum_{n=0}^{\infty} \frac{X^n}{n!} = EXP(X) \tag{4.5}$$

The limiting and rearrangement procedures leading to this result are valid not only for real and complex numbers, but for $n \times n$ matrices and bounded operators as well.

Example: We take an arbitrary vector X in the three-dimensional linear vector space of traceless 2×2 matrices spanned by the generators

X_a, X_b, X_c of $SL(2;R)$ given in (4.4)

$$X = aX_a + bX_b + cX_c = \left[\begin{array}{cc} a & b \\ c & -a \end{array} \right] \tag{4.6}$$

The exponential of this matrix is

$$EXP(X) = EXP(aX_a+bX_b+cX_c) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\begin{array}{cc} a & b \\ c & -a \end{array} \right]^n = I_2 \cosh\theta + X \frac{\sinh\theta}{\theta}$$

$$= \left[\begin{array}{cc} \cosh\theta + a\sinh(\theta)/\theta & b\sinh(\theta)/\theta \\ c\sinh(\theta)/\theta & \cosh\theta - a\sinh(\theta)/\theta \end{array} \right] \tag{4.7}$$

$$\theta^2 = a^2 + bc$$

The actual computation can be carried out either using brute force or finesse.

With brute force, each of the matrices X^n is computed explicitly, a pattern is recognized, and the sum is carried out. The first few powers are $X^0 = I_2$, $X^1 = X$ [given in (4.6)], and $X^2 = \theta^2 I_2$. Since X^2 is a multiple of the identity, $X^3 = X^2 X^1$ must be proportional to X $(= \theta^2 X)$, X^4 is proportional to the identity, and so on.

Finesse involves use of the Cayley-Hamilton theorem, that every matrix satisfies its secular equation. This means that a 2×2 matrix must satisfy a polynomial equation of degree 2. Thus we can replace X^2 by a function of $X^0 = I_2$ and $X^1 = X$. Similarly, X^3 can be replaced by a linear combination of X^2 and X, and then X^2 replaced by I_2 and X. By induction, any function of the 2×2 matrix X can be written in the form

$$F(X) = f_0(a,b,c)X^0 + f_1(a,b,c)X^1 \tag{4.8}$$

Furthermore, the functions f_0, f_1 are not arbitrary functions of the three parameters (a, b, c), but rather functions of the invariants of the matrix X. These invariants are the coefficients of the secular equation. The only such invariant for the 2×2 matrix X is $\theta^2 = a^2 + bc$. As a result, we know from general and simple considerations that

$$EXP(X) = f_0(\theta^2)I_2 + f_1(\theta^2)X \tag{4.9}$$

The two functions are $f_0(\theta^2) = 1 + \theta^2/2! + \theta^4/4! + \theta^6/6! + \cdots = \cosh\theta$ and $f_1(\theta^2) = 1+\theta^2/3!+\theta^4/5!+\theta^6/7!+\cdots = \sinh(\theta)/\theta$. These arguments are applicable to the exponential of any matrix Lie algebra.

The EXPonential operation provides a natural parameterization of the Lie group in terms of linear quantities. This function maps the linear vector space — the Lie algebra — to the geometric manifold that parameterizes the Lie group. We can expect to find a lot of geometry in the EXPonential map.

Three important questions arise about the reversibility of the process represented by

$$\text{Lie group} \underset{\text{EXP}}{\overset{ln}{\rightleftharpoons}} \text{Lie algebra} \tag{4.10}$$

(i) Does the EXPonential function map the Lie algebra back onto the entire Lie group?
(ii) Are Lie groups with isomorphic Lie algebras themselves isomorphic?
(iii) Is the mapping from the Lie algebra to the Lie group unique, or are there other ways to parameterize a Lie group?

These are very important questions. In brief, the answer to each of these questions is 'No.' However, as is very often the case, exploring the reasons for the negative result often produces more insight than a simple 'Yes' response would have. They will be treated in more detail in Chapter 7.

4.4 Properties of a Lie Algebra

We now turn to the properties of a Lie algebra. These are derived from the properties of a Lie group. A Lie algebra has three properties:

(i) The operators in a Lie algebra form a linear vector space;
(ii) The operators close under commutation: the commutator of two operators is in the Lie algebra;
(iii) The operators satisfy the Jacobi identity.

If X and Y are elements in the Lie algebra, then $g_1 = I + \epsilon X$ is an element in the Lie group near the identity for ϵ sufficiently small. In fact, so also is $I + \epsilon\alpha X$ for any real number α. We can form the product

$$(I + \epsilon\alpha X)(I + \epsilon\beta X) = I + \epsilon(\alpha X + \beta Y) + \text{higher order terms} \tag{4.11}$$

If X and Y are in the Lie algebra, then so is any linear combination of X and Y. The Lie algebra is therefore a linear vector space.

The commutator of two group elements is a group element:

$$\text{commutator of } g_1 \text{ and } g_2 = g_1 g_2 g_1^{-1} g_2^{-1} \tag{4.12}$$

If X and Y are in the Lie algebra, then for any ϵ, δ sufficiently small, $g_1(\epsilon) = EXP(\epsilon X)$ and $g_1(\epsilon)^{-1} = EXP(-\epsilon X)$ are group elements near the identity, as are $g_2(\delta)^{\pm 1} = EXP(\pm\delta Y)$. Expanding the commutator to lowest order nonvanishing terms, we find

$$EXP(\epsilon X)EXP(\delta Y)EXP(-\epsilon X)EXP(-\delta Y) =$$

$$I + \epsilon\delta\,(XY - YX) = I + \epsilon\delta\,[X, Y] \tag{4.13}$$

Therefore, the commutator of two group elements, $g_1(\epsilon) = EXP(\epsilon X)$ and $g_2(\delta) = EXP(\delta Y)$, which is in the group G, requires the commutator of the operators X and Y, $[X, Y] = (XY - YX)$, to be in its Lie algebra $\mathfrak{g}$

$$g_1 g_2 g_1^{-1} g_2^{-1} \in G \Leftrightarrow [X, Y] \in \mathfrak{g} \tag{4.14}$$

The commutator (4.12) provides information about the structure of a group. If the group is commutative then the commutator in the group (4.12) is equal to the identity. The commutator in the algebra vanishes

$$g_1 g_2 g_1^{-1} g_2^{-1} = I \Rightarrow [X, Y] = 0 \tag{4.15}$$

If H is an invariant subgroup of G, then $g_1 H g_1^{-1} \subset H$. This means that if X is in the Lie algebra of G and Y is in the Lie algebra of H

$$g_1 H g_1^{-1} \in H \Rightarrow [X, Y] \in \text{Lie algebra of } H \tag{4.16}$$

If X, Y, Z are in the Lie algebra, then the **Jacobi identity** is satisfied

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{4.17}$$

This identity involves the cyclic permutation of the operators in a double commutator. For matrices this identity can be proved by opening up the commutators ($[X, Y] = XY - YX$) and showing that the 12 terms so obtained cancel pairwise. This proof remains true when the operators X, Y, Z are not matrices but operators for which composition (e.g. XY is well-defined, as are all other pairwise products) is defined. When operator products (as opposed to commutators) are not defined, this method of proof fails but the theorem (it is *not* an identity) remains true. This theorem represents an integrability condition on the functions that define the group multiplication operation on the underlying manifold.

To summarize, a Lie algebra $\mathfrak{g}$ has the following structure:

(i) It is a linear vector space under vector addition and scalar multiplication. If $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$ then every linear combination of X and Y is in $\mathfrak{g}$.

$$X \in \mathfrak{g}, \quad Y \in \mathfrak{g}, \quad \alpha X + \beta Y \in \mathfrak{g}$$

(ii) It is an algebra under commutation. If $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$ then their commutator is in $\mathfrak{g}$.

$$X \in \mathfrak{g}, \quad Y \in \mathfrak{g}, \quad [X, Y] \in \mathfrak{g}$$

This property is called 'closure under commutation.'

(iii) The Jacobi identity is satisfied. If $X \in \mathfrak{g}$, $Y \in \mathfrak{g}$, and $Z \in \mathfrak{g}$, then

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Example: The three generators (4.4) of the Lie group $SL(2; R)$ obey the commutation relations

$$\begin{array}{rcr} [X_a, X_b] & = & 2X_b \\ [X_a, X_c] & = & -2X_c \\ [X_b, X_c] & = & X_a \end{array} \tag{4.18}$$

It is an easy matter to verify that the Jacobi identity is satisfied for this Lie algebra.

4.5 Structure Constants

Since a Lie algebra is a linear vector space we can introduce all the usual concepts of a linear vector space, such as dimension, basis, inner product. The dimension of the Lie algebra $\mathfrak{g}$ is equal to the dimension of the manifold that parameterizes the Lie group G. If the dimension is n, it is possible to choose n linearly independent vectors in the Lie algebra (a basis for the linear vector space) in terms of which any operator in $\mathfrak{g}$ can be expanded. If we call these basis vectors, or basis operators $X_1, X_2, \cdots, X_n$, then we can ask several additional questions such as: Is there a natural choice of basis vectors? Is there a reasonable definition of inner product (X_i, X_j)? We return to these questions shortly.

Since the linear vector space is closed under commutation, the commutator of any two basis vectors can be expressed as a linear superposition of basis vectors

$$[X_i, X_j] = C_{ij}{}^{k} X_k \tag{4.19}$$

The coefficients $C_{ij}{}^{k}$ in this expansion are called **structure constants**.

The structure of the Lie algebra is completely determined by its structure constants. The antisymmetry of the commutator induces a corresponding antisymmetry in the structure constants

$$[X_i, X_j] + [X_j, X_i] = 0 \qquad {C_{ij}}^k + {C_{ji}}^k = 0 \tag{4.20}$$

Under a change of basis transformation

$$X_i = {A_i}^r Y_r \tag{4.21}$$

the structure constants change in a systematic way

$${C'_{rs}}^t = (A^{-1})_r{}^i (A^{-1})_s{}^j {C_{ij}}^k {A_k}^t \tag{4.22}$$

(second order covariant, first order contravariant tensor). This piece of information is surprisingly useless.

Example: The only nonzero structure constants for the three basis vectors X_a, X_b, X_c (4.4) in the Lie algebra $\mathfrak{sl}(2;R)$ for the Lie group $SL(2;R)$ are, from (4.18)

$${C_{ab}}^b = -{C_{ba}}^b = +2, \qquad {C_{ac}}^c = -{C_{ca}}^c = -2, \qquad {C_{bc}}^a = -{C_{cb}}^a = +1 \tag{4.23}$$

4.6 Regular Representation

A better way to look at a change of basis transformation is to determine how the change of basis affects the commutator of an arbitrary element Z in the algebra

$$[Z, X_i] = R(Z)_i{}^j X_j \tag{4.24}$$

Under the change of basis (4.21) we find

$$[Z, Y_r] = S(Z)_r{}^s Y_s \tag{4.25}$$

where

$${S_r}^s(Z) = (A^{-1})_r{}^i R(Z)_i{}^j {A_j}^s \tag{4.26}$$

In this manner the effect of a change of basis on the structure constants is reduced to a study of similarity transformations.

The association of a matrix $R(Z)$ with each element of a Lie algebra is called the **Regular representation**

$$Z \quad \underset{\text{Representation}}{\overset{\text{Regular}}{\longrightarrow}} \quad R(Z) \tag{4.27}$$

The regular representation of an n-dimensional Lie algebra is a set of $n \times n$ matrices. This representation contains exactly as much information as the structure constants, for the regular representation of a basis vector is

$$[X_i, X_j] = R(X_i)_j{}^k X_k = C_{ij}{}^k X_k \tag{4.28}$$

so that

$$R(X_i)_j{}^k = C_{ij}{}^k \tag{4.29}$$

The regular representation is an extremely useful tool for resolving a number of problems.

Example: The regular representation of the Lie algebra $\mathfrak{sl}(2;R)$ is easily constructed, since the structure constants have been given in (4.23)

$$R(X) = R(aX_a+bX_b+cX_c) = aR(X_a)+bR(X_b)+cR(X_c) = \begin{bmatrix} 0 & -2b & 2c \\ -c & 2a & 0 \\ b & 0 & -2a \end{bmatrix} \tag{4.30}$$

The rows and columns of this 3×3 matrix are labeled by the indices a, b and c, respectively.

4.7 Structure of a Lie Algebra

The first step in the classification problem is to investigate the regular representation of the Lie algebra under a change of basis. We look for a choice of basis that brings the matrix representative of every element in the Lie algebra simultaneously to one of the three forms shown in Fig. 4.1. The first term (nonsemisimple, ...) is applied typically to Lie groups and algebras while the second term (reducible, ...) is typically applied to representations.

Example: It is not possible to simultaneously reduce the regular representatives of the three generators X_a, X_b, and X_c of $\mathfrak{sl}(2;R)$ to either the nonsemisimple or the semisimple form. This algebra is therefore simple. However, the Euclidean group $E(2)$ with structure

$$E(2) = \begin{bmatrix} \cos\theta & \sin\theta & t_1 \\ -\sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.31}$$

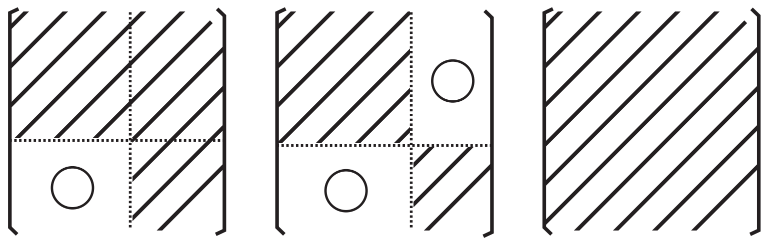

Fig. 4.1. Standard forms into which a representation can be reduced

has a Lie algebra with three infinitesimal generators

$$L_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.32}$$

and regular representation

$$R(\theta L_z + t_1 P_x + t_2 P_y) = \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ -t_2 & t_1 & 0 \end{bmatrix} \tag{4.33}$$

where the rows and columns are labeled successively by the basis vectors P_x, P_y, and L_z. This regular representation has the block diagonal structure of a nonsemisimple Lie algebra. The algebra, and the original group, are therefore nonsemisimple.

There is a beautiful structure theory for simple and semisimple Lie algebras. This will be discussed in Chapter 9. A structure theory exists for nonsemisimple Lie algebras. It is neither as beautiful nor as complete as the structure theory for simple Lie algebras.

4.8 Inner Product

Since a Lie algebra is a linear vector space, we are at liberty to impose on it all the structures that make linear vector spaces so simple and convenient to use. These include inner products and appropriate choices of basis vectors.

Inner products in spaces of matrices are simple to construct. A well-known and very useful inner product when A, B are $p \times q$ matrices is the **Hilbert-Schmidt** inner product

$$(A,B) = \mathrm{Tr}\ A^{\dagger}B \tag{4.34}$$

This inner product is positive definite, that is

$$(A,A) = \sum_i \sum_j |A_i{}^j|^2 \geq 0, \qquad = 0 \Rightarrow A = 0 \tag{4.35}$$

If we were to adopt the Hilbert-Schmidt inner product on the regular representation of $\mathfrak{g}$, then

$$(X_i, X_j) = \mathrm{Tr}\ R(X_i)^{\dagger}R(X_j) = \sum_r \sum_s R(X_i)_r{}^{s^*} R(X_j)_r{}^s = \sum_r \sum_s C_{ir}{}^{s^*} C_{jr}{}^s \tag{4.36}$$

This inner product is positive semidefinite on $\mathfrak{g}$: it vanishes identically on those generators that commute with all operators in the Lie algebra (X_i, where $C_{i\star}{}^{*} = 0$) and also on all generators that are not representable as the commutator of two generators (X_i, where $C^i_{*\star} = 0$).

The Hilbert-Schmidt inner product is a reasonable choice of inner product from an algebraic point of view. However, there is an even more useful choice of inner product that provides both algebraic and geometric information. This is defined by

$$(X_i, X_j) = \mathrm{Tr}\ R(X_i)R(X_j) = \sum_r \sum_s R(X_i)_r{}^s R(X_j)_s{}^r = \sum_r \sum_s C_{ir}{}^s C_{js}{}^r \tag{4.37}$$

This inner product is called the **Cartan-Killing inner product**, or **Cartan-Killing form**. It is in general an indefinite inner product. It is used extensively in the classification theory of Lie algebras.

The Cartan-Killing metric can be used to advantage to make further refinements on the structure theory of a Lie algebra. The vector space of the Lie algebra can be divided into three subspaces under the Cartan-Killing inner product. The inner product is positive-definite, negative-definite, and identically zero on these three subspaces:

$$\mathfrak{g} = V_+ + V_- + V_0 \tag{4.38}$$

The subspace V_0 is a subalgebra of $\mathfrak{g}$. It is the largest nilpotent invariant subalgebra of $\mathfrak{g}$. Under exponentiation, this subspace maps onto the maximal nilpotent invariant subgroup in the original Lie group.

The subspace V_- is also a subalgebra of $\mathfrak{g}$. It consists of compact (a topological property) operators. That is to say, the exponential of this subspace is a subset of the original Lie group that is parameterized by a compact manifold. It also forms a subalgebra in $\mathfrak{g}$ (not invariant).

Finally, the subspace V_+ is not a subalgebra of $\mathfrak{g}$. It consists of noncompact operators. The exponential of this subspace is parameterized by a noncompact submanifold in the original Lie group.

In short, a Lie algebra has the following decomposition under the Cartan-Killing inner product

$$\mathfrak{g} \quad \begin{matrix} \text{Cartan} - \text{Killing} \\ \longrightarrow \\ \text{inner product} \end{matrix} \quad \begin{matrix} V_0 & \text{nilpotent invariant subalgebra} \\ V_- & \text{compact subalgebra} \\ V_+ & \text{noncompact operators} \end{matrix} \tag{4.39}$$

We return to the structure of Lie algebras in Chapter 8 and the classification of simple Lie algebras in Chapter 10.

Example: The Cartan-Killing inner product on the regular representation (4.30) of $\mathfrak{sl}(2;R)$ is

$$(X,X) = \text{Tr } R(X)R(X) = \text{Tr} \begin{bmatrix} 0 & -2b & 2c \\ -c & 2a & 0 \\ b & 0 & -2a \end{bmatrix}^2 = 8(a^2+bc) \tag{4.40}$$

From this we easily drive the form of the metric for the Cartan-Killing inner product:

$$8(a^2+bc) = \begin{pmatrix} a & b & c \end{pmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{4.41}$$

A convenient choice of basis vectors is one that diagonalizes this metric matrix: X_a and $X_\pm = X_b \pm X_c$. In this basis the metric matrix is

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \quad \begin{matrix} X_a \\ X_+ \\ X_- \end{matrix} \tag{4.42}$$

In this representation it is clear that the operator X_- spans a one-dimensional compact subalgebra in $\mathfrak{sl}(2;R)$ and the generators X_a, X_+ are noncompact.

We should point out here that the inner product can also be computed even more simply in the defining 2×2 matrix representation of $\mathfrak{sl}(2;R)$

$$(X,X) = \text{Tr} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = 2(a^2+bc) \tag{4.43}$$

This gives an inner product that is proportional to the inner product derived from the regular representation. This is not an accident, and this observation can be used to compute the Cartan-Killing inner products very rapidly for all matrix Lie algebras.

4.9 Invariant Metric and Measure on a Lie Group

The properties of a Lie algebra can be identified with the properties of the corresponding Lie group at the identity.

Once the properties of a Lie group have been determined in the neighborhood of the identity, these properties can be translated to the neighborhood of any other group operation. This is done by multiplying the identity and its neighborhood on the left (or right) by that group operation.

Two properties that are useful to define over the entire manifold are the **metric** and **measure**. We assume the coordinates of the identity are $(\alpha^1, \alpha^2, \cdots, \alpha^n)$ and the coordinates of a point near the identity are $(\alpha^1 + d\alpha^1, \alpha^2 + d\alpha^2, \cdots, \alpha^n + d\alpha^n)$. If $(x^1, x^2, \cdots, x^n)$ represents some other group operation, then the point $(\alpha^1 + d\alpha^1, \alpha^2 + d\alpha^2, \cdots, \alpha^n + d\alpha^n)$ is mapped to the point $(x^1 + dx^1, x^2 + dx^2, \cdots, x^n + dx^n)$ under left (right) multiplication by the group operation associated with $(x^1, x^2, \cdots, x^n)$. The displacements dx and $d\alpha$ are related by a position-dependent *linear* transformation

$$dx^r = M(x)^r{}_i d\alpha^i \tag{4.44}$$

Suppose now that the distance ds between the identity and a point with coordinates $\alpha^i + d\alpha^i$ infinitesimally close to the identity is given by

$$ds^2 = g_{ij}(Id) d\alpha^i d\alpha^j \tag{4.45}$$

Any metric can be chosen at the identity, but the most usual choice is the Cartan-Killing inner product. Can we define a metric at x, $g_{rs}(x)$, with the property that the arc length is an invariant?

$$g_{rs}(x) dx^r dx^s = g_{ij}(Id) d\alpha^i d\alpha^j \tag{4.46}$$

In order to enforce the invariance condition, the metric at x, $g(x)$, must be related to the metric at the identity by

$$g(x) = M^{-1}(x)^t g(Id) M^{-1}(x) \tag{4.47}$$

The volume elements at the identity and x are

$$\begin{aligned} dV(Id) &= d\alpha^1 \wedge d\alpha^2 \wedge \cdots \wedge d\alpha^n \\ dV(x) &= dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \| M \| \, d\alpha^1 \wedge d\alpha^2 \wedge \cdots \wedge d\alpha^n \end{aligned} \tag{4.48}$$

The two volume elements can be made equal by introducing a measure over the manifold and defining an invariant volume

$$d\mu(x) = \rho(x)dV(x) = \rho(x) \parallel M(x) \parallel dV(Id) \Rightarrow \rho(x) = \parallel M(x) \parallel^{-1} \tag{4.49}$$

Example: Under the simple parameterization (4.1) of the group $SL(2;R)$ the neighborhood of the identity is parameterized by (4.3). We move a neighborhood of the identity to the neighborhood of the group operation parameterized by (x, y, z) using left multiplication as follows

$$\begin{bmatrix} 1+x & y \\ z & \frac{1+yz}{1+x} \end{bmatrix} \times \begin{bmatrix} 1+d\alpha^1 & d\alpha^2 \\ d\alpha^3 & 1-d\alpha^1 \end{bmatrix} = \begin{bmatrix} 1+(x+dx) & y+dy \\ z+dz & \frac{1+(y+dy)(z+dz)}{1+(x+dx)} \end{bmatrix}$$

$$= \begin{bmatrix} (1+x)(1+d\alpha^1)+yd\alpha^3 & (1+x)d\alpha^2+y(1-d\alpha^1) \\ z(1+d\alpha^1)+\dfrac{(1+yz)d\alpha^3}{(1+x)} & zd\alpha^2+\dfrac{(1+yz)(1-d\alpha^1)}{(1+x)} \end{bmatrix} \tag{4.50}$$

The linear relation between the infinitesimals $(d\alpha^1, d\alpha^2, d\alpha^3)$ in the neighborhood of the identity and the infinitesimals (dx, dy, dz) in the neighborhood of the group operation (x, y, z) can now be read off, matrix element by matrix element

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 1+x & 0 & y \\ -y & 1+x & 0 \\ z & 0 & \frac{1+yz}{1+x} \end{bmatrix} \begin{bmatrix} d\alpha^1 \\ d\alpha^2 \\ d\alpha^3 \end{bmatrix} \tag{4.51}$$

From this linear transformation we immediately compute the invariant measure by taking the inverse of the determinant

$$d\mu(x) = \rho(x, y, z)dx \wedge dy \wedge dz = \frac{dx \wedge dy \wedge dz}{1+x} \tag{4.52}$$

The invariant metric is somewhat more difficult, as it involves computing the inverse of the linear transformation (4.51). The result is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{left translation} \\ \longrightarrow \\ \text{by } (x, y, z) \end{matrix} \begin{bmatrix} \dfrac{2(1+yz)}{(1+x)^2} & -\dfrac{z}{(1+x)} & -\dfrac{y}{(1+x)} \\ -\dfrac{z}{(1+x)} & 0 & 1 \\ -\dfrac{y}{(1+x)} & 1 & 0 \end{bmatrix} \tag{4.53}$$

The invariant measure (4.52) can be derived from the invariant metric (4.53) in the usual way (cf. Problem (4.11)).

4.10 Conclusion

The structure that results from the linearization of a Lie group is called a Lie algebra. Lie algebras are linear vector spaces. They are endowed with an additional combinatorial operation, the commutator $[X,Y] = (XY - YX)$, and obey the Jacobi identity. Since they are linear vector spaces, many powerful tools are available for their study. It is possible to define an inner product that reflects not only the algebraic properties of the original Lie group, but its topological properties as well. The properties of a Lie algebra can be identified with the properties of the parent Lie group in the neighborhood of the identity. These structures can be moved to neighborhoods of other points in the group manifold by a suitable group multiplication.

The linearization procedure is more or less invertible (a little less than more). The inversion is carried out by the EXPonential mapping.

4.11 Problems

1. Carry out the commutator calculation for $g_1 = (I + \epsilon X)$, $g_1^{-1} = (I + \epsilon X)^{-1} = I - \epsilon X + \epsilon^2 X^2 - \cdots$, with similar expressions for g_2, to obtain the same result as in (4.13). In other words, this local result is independent of the parameterization in the neighborhood of the identity.

2. The inner product of two vectors X and X' in a linear vector space can be computed if the inner product of a vector with itself is known. This is done by the method of **polarization**. For a real linear vector space the argument is as follows:

$$(X', X) = \frac{1}{2}\left[(X' + X, X' + X) - (X', X') - (X, X)\right]$$

a. Verify this.
b. Extend to complex linear vector spaces.
c. Use the result from Eq. (4.43) that $(X, X) = 2(a^2 + bc)$ to show $(X', X) = 2a'a + b'c + c'b$.

3. Suppose that the $n \times n$ matrix Y is defined as the exponential of an $n \times n$ matrix X in a Lie algebra: $Y = e^X$. Show that "for Y sufficiently close to the identity" the matrix X can be expressed as

$$X = -\sum_{n=1}^{\infty} \frac{(I - Y)^n}{n} \qquad (4.54)$$

Show that this expansion converges when X and Y are symmetric if the real eigenvalues λ_i of Y all satisfy $0 < \lambda_i < +2$. Show that if $Y \in SL(2;R)$ and tr $Y < -2$ this expansion does not converge. That is, there is no 2×2 matrix $X \in \mathfrak{sl}(2;R)$ with the property tr $e^X < -2$.

4. The Lie algebra of $SO(3)$ is spanned by three 3×3 antisymmetric matrices $\mathbf{L} = (L_1, L_2, L_3) = (X_{23}, X_{31}, X_{12})$, with

$$\theta \cdot \mathbf{L} = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \theta_{12} & \theta_{13} \\ \theta_{21} & 0 & \theta_{23} \\ \theta_{31} & \theta_{32} & 0 \end{bmatrix} = \mathbf{X} \qquad (4.55)$$

Use the Cayley-Hamilton theorem to show

$$e^{\theta \cdot \mathbf{L}} = I_3 f_0(\theta) + \mathbf{X} f_1(\theta) + \mathbf{X}^2 f_2(\theta) \qquad (4.56)$$

where θ is the single invariant that can be constructed from the matrix $\mathbf{X} = \theta \cdot \mathbf{L}$: $\theta^2 = \theta_1^2 + \theta_2^2 + \theta_3^2$. Show

$$\begin{array}{rclcrcl} f_0(\theta) & = & \cos\theta & \text{or} & f_0(\theta) & = & \cos\theta \\ f_1(\theta) & = & \sin(\theta)/\theta & & \theta f_1(\theta) & = & \sin(\theta) \\ f_2(\theta) & = & (1-\cos(\theta))/\theta^2 & & \theta^2 f_2(\theta) & = & 1-\cos(\theta) \end{array}$$

5. The Lie algebra for the matrix group $SO(n)$ consists of antisymmetric $n \times n$ matrices. Show that a useful set of basis vectors (matrices) consists of the $n(n-1)/2$ matrices $X_{ij} = -X_{ji}$ $(1 \leq i \neq j \leq n)$ with matrix elements $(X_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}$.

a. Show that these matrices satisfy the commutation relations

$$[X_{ij}, X_{rs}] = X_{is}\delta_{jr} + X_{jr}\delta_{is} - X_{ir}\delta_{js} - X_{js}\delta_{ir} \qquad (4.57)$$

b. Show that the operators $\mathcal{X}_{ij} = x^i\partial_j - x^j\partial_i$ satisfy isomorphic commutation relations.

c. Show that bilinear products of boson creation and annihilation operators $\mathcal{B}_{ij} = b_i^\dagger b_j - b_j^\dagger b_i$ $(1 \leq i \neq j \leq n)$ satisfy isomorphic commutation relations.

c. Show that bilinear products of fermion creation and annihilation operators $\mathcal{F}_{ij} = f_i^\dagger f_j - f_j^\dagger f_i$ $(1 \leq i \neq j \leq n)$ satisfy isomorphic commutation relations.

6. The Jacobi identity for operators D, Y, Z (replace $X \to D$ in Eq. 4.17) can be rewritten in the form

$$[D,[Y,Z]] = [[D,Y],Z] + [Y,[D,Z]] \tag{4.58}$$

Show this. Compare with the expression for the differential operator

$$d(f \wedge g) = (df) \wedge g + f \wedge (dg)$$

It is for this reason that the Jacobi identity is sometimes called a differential identity.

7. For the matrix Lie algebra $\mathfrak{so}(4)$ the defining matrix representation consists of 4×4 antisymmetric matrices while the regular representation consists of 6×6 antisymmetric matrices. Construct the defining and regular matrix representations for the element $a_{ij}X_{ij}$ in the Lie algebra:

$$X = \sum_{ij} a_{ij}X_{ij} \to \begin{array}{rcl} \mathfrak{def}(X) & = & \sum a_{ij}\mathfrak{def}(X_{ij}) \\ \mathfrak{reg}(X) & = & \sum a_{ij}\mathfrak{reg}(X_{ij}) \end{array} \tag{4.59}$$

Construct the Cartan-Killing inner product using these two different matrix representations:

$$\mathrm{tr}\ \mathfrak{def}(X)\mathfrak{def}(X) \leftarrow (X,X) \to \mathrm{tr}\ \mathfrak{reg}(X)\mathfrak{reg}(X) \tag{4.60}$$

Show that the two inner products are proportional. What is the proportionality constant? How does this result extend to $SO(n)$? to $SO(p,q)$? Is there a simple relation between the proportionality constant and the dimensions of the defining and regular representations?

8. Assume a Lie algebra of $n \times n$ matrices is noncompact and its Cartan-Killing form splits this Lie algebra into three subspaces:

$$\mathfrak{g} \to V_0 + V_- + V_+$$

Show that the subspace V_- exponentiates onto a compact manifold. Do this by showing that the basis matrices in V_- have eigenvalues that are imaginary or zero, so that $EXP(V_-)$ is multiply periodic. Apply this construction to the noncompact groups $SO(3,1)$ and $SO(2,2)$. Show $EXP(V_-)$ is a two-sphere S^2 for $SO(3,1)$ and a two-torus T^2 for $SO(2,2)$.

9. Construct the infinitesimal generators for the group $SO(3)$ using the parameterizations proposed in Problems 13 and 15 in Chapter 3.

10. Use the exponential parameterization of $SO(3)$ to construct the linear transformation M (Eq. 4.44) describing displacements from the identity to displacements at the group operator $e^{\theta \cdot \mathbf{L}} \in SO(3)$. From this construct the invariant density $\rho(\theta)$ and the metric tensor $g_{\mu\nu}(\theta)$. Give a reason for the strange behavior (singularities) that these invariant quantities exhibit.

11. Compute the determinant of the metric tensor (4.53) on the group $SL(2;R)$. Show that the square root of the determinant is equal to the measure, in accordance with the standard result of Riemannian geometry that $dV(x) = \| g(x) \|^{1/2} d^n x$. Discuss the additional factors of 2 and -1 that appear in this calculation.

12. An inner product is $(\mathbf{x}, \mathbf{x})$ is imposed on a real n-dimensional linear vector space. It is represented by a real symmetric nonsingular $n \times n$ matrix $g_{rs} = (\mathbf{e}_r, \mathbf{e}_s)$, where $\mathbf{x} = \mathbf{e}_i x^i$. The inverse matrix, g^{rs}, is well defined.

a. Lie group G preserves inner products. If $\mathbf{y} = G\mathbf{x}$, $(\mathbf{y}, \mathbf{y}) = (\mathbf{x}, \mathbf{x})$. Show $G^t g G = g$.
b. Show the Lie algebra H of G satisfies $H^t g + gH = 0$.
c. Show that the infinitesimal generators of G are $X_{rs} = g_{rt} x^t \partial_s - g_{st} x^t \partial_r$.
d. Show that the operators X_{rs} satisfy commutation relations

$$[X_{ij}, X_{rs}] = +X_{is} g_{jr} + X_{jr} g_{is} - X_{ir} g_{js} + X_{js} g_{ir}$$

13. Every real unimodular 2×2 matrix M can be written in the form $M = SO$, where S is a real symmetric unimodular matrix and O is a real orthogonal matrix.

In Group		Relation	In Algebra	
$S^t = S^{+1}$	$\det(S) = +1$	$S = e^{\Sigma}$	$\mathrm{Tr}\ \Sigma = 0$	$\Sigma^t = +\Sigma$
$O^t = O^{-1}$	$\det(O) = +1$	$O = e^{A}$	$\mathrm{Tr}\ A = 0$	$A^t = -A$

a. Show that $MM^t = S^2 = e^{2\Sigma}$.
b. Show $O = S^{-1}M = e^{-\Sigma}M$
c. Write S as a power series expansion in Σ.
d. Write Σ as a power series expansion in $S - I_2$.
Under what conditions are these expansions valid?

14. Extend the result of the previous problem to complex $n \times n$ matrices $M = HU$, with M arbitrary but nonsingular, $H^\dagger = H^{+1}$ hermitian and $U^\dagger = U^{-1}$ unitary.

15. Transfer matrices have been described in Chapter 3, Problem 24. In one dimension the transfer matrix for a scattering potential, with free particles incident from the left or right with momentum $\hbar k_L$ or $-\hbar k_R$, has the form [34]

$$\left[\begin{array}{cc} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ \beta_R - i\beta_I & \alpha_R - i\alpha_I \end{array}\right] \tag{4.61a}$$

The matrix elements are given explicitly by

$$\begin{array}{ll} 2\alpha_R = +m_{11} + \dfrac{k_R}{k_L} m_{22} & 2\alpha_I = +m_{12}k_R - k_L^{-1} m_{21} \\ 2\beta_R = +m_{11} - \dfrac{k_R}{k_L} m_{22} & 2\beta_I = -m_{12}k_R - k_L^{-1} m_{21} \end{array} \tag{4.61b}$$

The real quantities m_{ij} are the four matrix elements of a group operation in $SL(2;R)$. They are energy dependent. By appropriate choice of $\hbar k_L = \hbar k_R$ and the matrix elements m_{ij}, construct three infinitesimal generators for the group of the transfer matrix for scattering states. Show that they are

$$\left[\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right] \qquad \left[\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right] \qquad \left[\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array}\right] \tag{4.61c}$$

Show that these three matrices span the Lie algebra of the group $SU(1,1)$.

16. The transfer matrix for a potential that possesses bound states has the form

$$\left[\begin{array}{cc} \alpha_1 + \alpha_2 & \beta_1 + \beta_2 \\ \beta_1 - \beta_2 & \alpha_1 - \alpha_2 \end{array}\right] \tag{4.62a}$$

The matrix elements are given explicitly by

$$\begin{array}{ll} 2\alpha_1 = +m_{11} + \dfrac{\kappa_R}{\kappa_L} m_{22} & 2\alpha_2 = -m_{12}\kappa_R - \kappa_L^{-1} m_{21} \\ 2\beta_1 = +m_{11} - \dfrac{\kappa_R}{\kappa_L} m_{22} & 2\beta_2 = +m_{12}\kappa_R - \kappa_L^{-1} m_{21} \end{array} \tag{4.62b}$$

The parameters κ_R and κ_L describe the decay length of the exponentially decaying wavefunction in the asymptotic left and right hand regions of the potential. The real quantities m_{ij} are the four matrix elements of a group operation in $SL(2;R)$. They are energy dependent. By appropriate choice of $\kappa_L = \kappa_R$ and the matrix elements m_{ij}, construct two infinitesimal generators for the group of the transfer matrix for bound states. Show that they are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.62c}$$

Show that these three matrices span the Lie algebra of the group $SL(2;R)$. Argue that there ought to be interesting relations (e.g., analytic continuations) between the scattering states (e.g., resonances) and bound states through the relation between the groups $SL(2;R)$ and $SU(1,1)$, which are isomorphic. How are the matrices (4.61a) and (4.62a), the matrix elements (4.61b) and (4,62b), and the infinitesimal generators (4.61c) and (4.62c) related to each other by analytic continuation? (Hint: $k_* = \sqrt{2m(E-V_*)/\hbar^2}$ for $E > V_*$ and $\kappa_* = \sqrt{2m(V_* - E)/\hbar^2}$ for $E < V_*$, $* = L, R$.)