Physics 428: Quantum Mechanics III
Prof. Michael S. Vogeley

Practice Problems 2

Problem 1

Consider the total spin of a two-particle system, \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \). Show that

\[
S^2 = \hbar^2 \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

(Hint: it might be helpful to use the identity \( \mathbf{S}_1 \cdot \mathbf{S}_2 = S_{1z}S_{2z} + \frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}). \))

Using the hint we get

\[
S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}
\]

All of the operators in this expression commute, so their order of operation on the kets does not matter. To speed up evaluation of the matrix elements \( \langle m_1' m_2' | S^2 | m_1 m_2 \rangle \), note that (I’ve used +, − as shorthand for \( m = \pm 1/2 \) here)

\[
S_1^2|m_1 m_2\rangle = S_2^2|m_1 m_2\rangle = \frac{3}{4}\hbar^2|m_1 m_2\rangle \quad (1)
\]
\[
S_{1z}S_{2z}|m_1 m_2\rangle = m_1 m_2 \hbar^2 |m_1 m_2\rangle \quad (2)
\]
\[
S_{1+} | m_2 \rangle = S_{2+} | m_1 \rangle = 0 \quad (3)
\]
\[
S_{1-} | m_2 \rangle = S_{2-} | m_1 \rangle = 0 \quad (4)
\]
\[
S_{1+} | m_2 \rangle = S_{2+} | m_1 \rangle = \hbar | m_1 m_2 \rangle \quad (5)
\]
\[
S_{1-} | m_2 \rangle = S_{2-} | m_1 \rangle = \hbar | m_1 m_2 \rangle \quad (6)
\]

I’ll leave out the factor of \( \hbar^2 \) in the following, then multiply it in at the end. The matrix element \( \langle + | S^2 | + \rangle \) gets 3/4 from \( S_1^2 \), 3/4 from \( S_2^2 \), 1/2 from \( 2S_{1z}S_{2z} \), 0 from \( S_{1+}S_{2-} \) because you can’t raise \( m_1 = 1/2 \), 0 from \( S_{1-}S_{2+} \) because you can’t raise \( m_2 = 1/2 \). These sum to 2, thus \( \langle + | S^2 | + \rangle = 2\hbar^2 \). Likewise, the \( \langle - | S^2 | - \rangle \) element gets 3/4 + 3/4 + 1/2 + 0 + 0 = 2, where the last two are zero because you can’t lower \( m_1 = -1/2 \) or \( m_2 = -1/2 \). \( \langle + | S^2 | - \rangle \) gets 3/4 + 3/4 − 1/2 + 0 = 1. The opposite spins give the −1/2, the first zero because you can’t raise \( m_1 = 1/2 \) or lower \( m_2 = -1/2 \) and \( \langle - | - \rangle = 0 \). \( \langle + | S^2 | + \rangle = \langle - | S^2 | - \rangle \) by symmetry (since the operators all commute). The element \( \langle + | S^2 | - \rangle \) gets 0 + 0 + 0 + 1 + 0, because \( \langle + | - \rangle = 0 \), thus the only non-zero term is from \( S_{1+}S_{2-} \) which yields \( \langle + | + \rangle = 1 \). \( \langle - | S^2 | + \rangle = \langle - | S^2 | - \rangle \) by symmetry. All of the other matrix elements are zero because \( \langle m_1' m_2' | m_1 m_2 \rangle = \delta_{m_1' m_1} \delta_{m_2' m_2} \).

Problem 2
Show that the eigenvectors of $\hat{n} \cdot \vec{\sigma}$ are
\[
\begin{pmatrix}
\cos(\theta/2)e^{-i\phi/2} \\
\sin(\theta/2)e^{i\phi/2}
\end{pmatrix}, \quad \begin{pmatrix}
-\sin(\theta/2)e^{-i\phi/2} \\
cos(\theta/2)e^{i\phi/2}
\end{pmatrix}
\]
where $\hat{n}$ is a unit vector in an arbitrary direction and $\vec{\sigma}$ is the vector of spin matrices.

Write the unit vector as $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, multiply by the Pauli matrices and add them up to show that the matrix is
\[
\hat{n} \cdot \vec{\sigma} = \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & -\cos \theta
\end{pmatrix}
\]
where we used $\cos \phi + i \sin \phi = e^{i\phi}$. The eigenvalues are the roots of
\[-\cos^2 \theta - \sin^2 \theta e^{i\phi} e^{-i\phi} + \lambda^2 = 0\]
thus $\lambda = \pm 1$.

To get the $\lambda = 1$ eigenvector, solve
\[
\begin{pmatrix}
\cos \theta - 1 & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & -\cos \theta - 1
\end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
This yields two equations, only one of which is independent, so we’ll write down only the first,
\[(\cos \theta - 1)a_1 + \sin \theta e^{-i\phi}b_1 = 0\]
or
\[
\frac{a_1}{b_1} = \frac{\sin \theta e^{-i\phi}}{1 - \cos \theta}
\]
This simplifies considerably if we apply the half-angle identities,
\[
\begin{alignat}{2}
sin \theta &= 2 \sin(\theta/2) \cos(\theta/2) \quad & (7) \\
cos \theta &= 1 - 2 \sin^2(\theta/2) \quad & (8)
\end{alignat}
\]
thus
\[
\frac{a_1}{b_1} = \frac{\cos(\theta/2)}{\sin(\theta/2)} e^{-i\phi}
\]
The column vector
\[
\begin{pmatrix}
\cos(\theta/2)e^{-i\phi/2} \\
\sin(\theta/2)e^{i\phi/2}
\end{pmatrix}
\]
is a possibility because it has norm $= 1$. This vector times any factor with modulus unity would also be a solution. We split the $e^{-i\phi}$ factor between the two elements just for symmetry (there’s no reason why only one of them should have an imaginary part). The $\lambda = -1$ eigenvector can be found by similar means, with the equation
\[
\frac{a_2}{b_2} = \frac{-\sin(\theta/2)}{\cos(\theta/2)} e^{-i\phi}
\]
suggesting the solution
\[
\begin{pmatrix}
\sin(\theta/2)e^{-i\phi/2} \\
\cos(\theta/2)e^{i\phi/2}
\end{pmatrix}
\]

**Problem 3**

A beam of particles of spin $\hbar/2$ is sent through a Stern-Gerlach apparatus, which divides the incident beam into two spatially separated components depending on the quantum numbers $m_s$ of the particles. One of the resulting beams is removed and the other beam is sent through a similar apparatus, the magnetic field of which is inclined by an angle $\theta$ with respect to the first. Again, the incoming beam is split into two components. What are the relative numbers of particles that appear in the two beams leaving the second apparatus? Derive the result using the Pauli spin formalism.

Let the magnetic field of the first apparatus lie along the $\hat{z}$ direction, thus $\mathbf{B} = B_0\hat{z}$. The second apparatus will be rotated by an angle $\theta$ from the first. The Hamiltonian of the first S-G is

\[ H = -\mu \cdot \mathbf{B} = \mu_B \sigma \cdot \mathbf{B} = \mu_B \sigma_z B_0 \]

If a particle enters the first S-G apparatus in a mixed state of eigenvalues of $\sigma_z$,

\[ \alpha \chi_+ + \beta \chi_- = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

then it exits the S-G in a pure up $\chi_+$ or down $\chi_-$ state. Recall that the relative probabilities of these two outgoing states is $|\alpha|^2/|\beta|^2$. Let’s block off the outgoing beam of the spin down particles. Thus, only particles in the $\chi_+$ state enter the second S-G apparatus. The magnetic field is rotated by $\theta$. Let the particle travel in the $\hat{y}$ direction and let the magnetic field rotate in the $x-z$ plane so that the magnetic field points in the direction $\hat{n} = (\sin \theta, 0, \cos \theta)$. Thus, the second Hamiltonian is

\[ H = -\mu \cdot \mathbf{B} = \mu_B \hat{n} \cdot \mathbf{B} = \mu_B B_0[\sin \theta \sigma_x + \cos \theta \sigma_z] \]

To determine the relative numbers of particles in the two beams that come out of the second S-G, we have to decompose the states that enter the S-G into eigenstates of the Hamiltonian, which are proportional to eigenvectors of

\[ \sin \theta \sigma_x + \cos \theta \sigma_z = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \]

Solving the characteristic equation, we find that this matrix has eigenvalues $\pm 1$ and eigenvectors

\[ \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \]

(See your result from Problem 2: these are the eigenvectors of $\hat{n} \cdot \sigma$, with $\phi = 0$.) Now decompose the incoming eigenvector into this basis,

\[ \chi_{\text{incoming}} = c_+ \chi_+ + c_- \chi_- \hat{n} \]
The coefficients are just the dot products
\[ c_+ = \chi_{\text{incoming}} \cdot \chi_+ \hat{n} = \cos(\theta/2) \]  \hspace{1cm} (9)
\[ c_- = \chi_{\text{incoming}} \cdot \chi_- \hat{n} = -\sin(\theta/2) \]  \hspace{1cm} (10)

The relative probability of the up and down states is then
\[
\frac{|c_+|^2}{|c_-|^2} = \frac{\cos^2(\theta/2)}{\sin^2(\theta/2)} = \cot^2(\theta/2)
\]

**Problem 4**

Assume that a one-particle system has an orbital angular momentum with a \( \hat{z} \) component of \( m \bar{\hbar} \) and a squared magnitude of \( l(l+1)\bar{\hbar}^2 \).

a) Show that
\[
\langle L_x \rangle = \langle L_y \rangle = 0
\]
(Hint: express these operators in term of the ladder operators \( L_+, L_- \).)

Recall that
\[
L_\pm |lm\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} |l, m \pm 1\rangle
\]
Therefore
\[
\langle lm|L_\pm|lm\rangle \propto \langle lm|l, m \pm 1\rangle = 0
\]
Writing \( L_x \) and \( L_y \) in terms of the ladder operators,
\[
\langle L_x \rangle = \langle lm|\frac{L_+ + L_-}{2}|lm\rangle = 0
\]
and
\[
\langle L_y \rangle = \langle lm|\frac{L_+ - L_-}{2}|lm\rangle = 0
\]

b) Show that
\[
\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{l(l+1)\hbar^2 - m^2\hbar^2}{2}
\]

Again, express the operators in terms of the ladder operators. Thus,
\[
L_x^2 = \frac{1}{4}(L_+ + L_-)^2 = \frac{1}{4}(L_+^2 + L_-^2 + L_+L_- + L_-L_+)
\]
Now take the expectation value. We know from part (a) that the expectation values of the first two terms are zero. For the third and fourth term we need to evaluate
\[
\langle lm|L_+L_-|lm\rangle = \hbar^2[l(l+1) - m(m-1)]
\]
and
\[\langle lm|L_-L_+|lm\rangle = h^2[l(l + 1) - m(m + 1)]\]
Substituting in above,
\[\langle L_x^2\rangle = \frac{1}{4}\langle lm|L_+L_- + L_-L_+|lm\rangle = \frac{1}{2}[l(l + 1) - m(m + 1)]\]

\textbf{c)} A measurement of a component of angular momentum making an angle \(\theta\) with respect to \(\hat{z}\) is made. Calculate the mean value and the mean square value for this measurement.

Express the operator \(L_\theta\) in terms of \(L_x, L_y, L_z\),
\[L_\theta = L_x \sin \theta \cos \phi L_y \sin \theta \sin \phi + L_z \cos \theta\]
Notice that we can write the \(L_x, L_y\) operators in terms of the ladder operators \(L_+, L_-\), thus
\[L_\theta = \frac{1}{2} L_+ \sin \theta e^{-i\phi} + L_- \sin \theta e^{i\phi} + L_z \cos \theta\]
We'll evaluate \(\langle lm|L_\theta^2|lm\rangle\) in two steps, breaking up the expectation value in an operator acting on the bra and the ket separately, then multiplying together, \(\langle lm|L_\theta^2|lm\rangle = \langle lm|L_\theta L_\theta|lm\rangle\). First,
\[L_\theta|lm\rangle = \frac{1}{2} \sin \theta e^{-i\phi} L_+|lm\rangle + \frac{1}{2} \sin \theta e^{i\phi} |lm\rangle + \cos \theta L_z|lm\rangle\]  
\[= \frac{1}{2} \sin \theta e^{-i\phi} \sqrt{l(l + 1) - m(m + 1)}|lm\rangle\]  
\[+ \frac{1}{2} \sin \theta e^{i\phi} \sqrt{l(l + 1) - m(m - 1)}|lm\rangle\]  
\[+ \cos \theta m|lm\rangle\]
Second part is operation on the left side on the bra,
\[\langle lm|L_\theta\]
which has the same result as the ket, except for complex conjugation, which flips the signs of the \(e^{i\phi}\) factors. Thus, those factors cancel out. Multiply together, factor all the constants, and group terms, we get
\[\langle L_\theta^2\rangle = \frac{1}{2} h^2[l(l + 1) - m^2] \sin^2 \theta + m^2 h^2 \cos^2 \theta\]

\textbf{Problem 5}

Write down the ladder operators \(S_+\) and \(S_-\) in matrix form.
Recall $S_+ = S_x + iS_y$, $S_- = S_x - iS_y$, and

$$S_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Thus,

$$S_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
Problem 6

Suppose that we add the orbital angular momenta of two particles, both with \( l = 1 \). Express the \(|lm\rangle\) kets \(|2,0\rangle\) and \(|2,-1\rangle\) of the total-l basis as linear combinations of product basis kets \(|m_1m_2\rangle\).

First, recall our example from class, in which we lowered \(|2,2\rangle\) to create the state \(|2,1\rangle\) from the product basis using the lowering operator, whose action is

\[
L_-|lm\rangle = \hbar \sqrt{l(l+1) - m(m-1)|l, m-1}\rangle
\]

Recall that we operated on both sides of the state

\[|l = 2, m = 2\rangle = |m_1 = 1, m_2 = 1\rangle\]

and used \(L_- = L_{1-} + L_{2-}\) to find

\[|2,1\rangle = 2^{-1/2}(|0,1\rangle + |1,0\rangle)\]

Now do the same to get the \(|2,1\rangle\) and \(|2,0\rangle\). First operate on the \(|lm\rangle\) ket on the left hand side above,

\[L_-|2,1\rangle = \sqrt{6}\hbar|2,0\rangle\]

then on the \(|m_1m_2\rangle\) kets on the right hand side,

\[
\begin{align*}
L_-|0,1\rangle &= \sqrt{2}\hbar|0,1\rangle + \sqrt{2}\hbar|0,0\rangle \\
L_-|1,0\rangle &= \sqrt{2}\hbar|0,0\rangle + \sqrt{2}|1,-1\rangle
\end{align*}
\]

Not forgetting the \(2^{-1/2}\) factor that multiplies the product basis kets, the total-l basis ket is

\[|2,0\rangle = 6^{-1/2}|2|0,0\rangle + |1,-1\rangle + |1,1\rangle\]

where the kets on the right side are in the product basis. Check that \(\langle 2,0|2,0\rangle = 6^{-1/2}(4 + 1 + 1) = 1\), thus the ket is properly normalized. Now proceed to \(|2,-1\rangle\) by lowering \(|2,0\rangle\),

\[L_-|2,0\rangle = \sqrt{6}\hbar|2,-1\rangle\]

and lowering the product basis kets,

\[
\begin{align*}
L_-|0,0\rangle &= \sqrt{2}\hbar|0,0\rangle + \sqrt{2}|0,-1\rangle \\
L_-|1,-1\rangle &= \sqrt{2}|0,-1\rangle + 0 \\
L_-|--1,1\rangle &= 0 + \sqrt{2}|1,0\rangle
\end{align*}
\]

Thus the total-l basis ket is

\[
\begin{align*}
|2,-1\rangle &= \frac{1}{6}[2(\sqrt{2}|0,0\rangle + \sqrt{2}|0,-1\rangle) + \sqrt{2}|0,-1\rangle + \sqrt{2}|1,0\rangle] \\
&= 2^{-1/2}(1\langle 0,0\rangle + |0,-1\rangle)
\end{align*}
\]

Again, check that \(\langle 2,-1|2,-1\rangle = (1/2)(2) = 1\).