Problem 1: 1-D Scattering and Phase Shift

A particle of mass \( m \) and energy \( E \) moves in one dimension in the positive \( x \) direction. It encounters a potential of the form

\[
V(x) = \begin{cases} 
0, & x < -a \\
-V_0, & -a \leq x \leq 0 \\
\infty, & x > 0
\end{cases}
\]

a. If the incoming wave is \( \psi_i(x) = A e^{ikx} \) (where \( k = \sqrt{2meE} \)), find the reflected wave. You should get the answer

\[
\psi_R(x) = A e^{-2ikx} \left[ \frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)} \right] e^{-ikx}
\]

where \( k' = \sqrt{2m(E + V_0)/\hbar} \).

b. Confirm that the reflected wave has the same amplitude as the incident wave.

c. Find the phase shift \( \delta \) (see Griffiths Eq. 11.40 for the definition) for the case of a very deep well \( E \ll V_0 \). You should get the answer \( \delta = -ka \).

(a) In the region to the left

\[
\psi(x) = A e^{ikx} + B e^{-ikx} \quad (x \leq -a).
\]

In the region \(-a < x < 0\), the Schrödinger equation gives

\[
-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - V_0\psi = E\psi \quad \Rightarrow \quad \frac{d^2\psi}{dx^2} = -k'\psi,
\]

where \( k' = \sqrt{2m(E + V_0)/\hbar} \). The general solution is

\[
\psi = C \sin(k'x) + D \cos(k'x)
\]

But \( \psi(0) = 0 \) implies \( D = 0 \), so

\[
\psi(x) = C \sin(k'x) \quad (-a \leq x \leq 0).
\]

The continuity of \( \psi(x) \) and \( \psi'(x) \) at \( x = -a \) says

\[
A e^{-ika} + B e^{ika} = -C \sin(k'a), \quad ik A e^{-ika} - ik B e^{ika} = k' C \cos(k'a).
\]

Divide and solve for \( B \):

\[
\frac{ik A e^{-ika} - ik B e^{ika}}{A e^{-ika} + B e^{ika}} = -k' \cot(k'a),
\]

\[
A e^{-ika} - ik B e^{ika} = -A e^{-ika} k' \cot(k'a) - B e^{ika} k' \cot(k'a).
\]

\[
B e^{ika} [-ik + k' \cot(k'a)] = A e^{-ika} [-ik - k' \cot(k'a)]
\]

\[
B = A e^{-2ika} \left[ \frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)} \right].
\]
(b) 

$$|B|^2 = |A|^2 \left[ \frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)} \right] \cdot \left[ \frac{k + ik' \cot(k'a)}{k - ik' \cot(k'a)} \right] = |A|^2. \quad \checkmark$$

(c) From part (a) the wave function for $x < -a$ is

$$\psi(x) = Ae^{ikx} + Ae^{-2i\alpha} \left[ \frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)} \right] e^{-ikx}.$$ 

But by definition of the phase shift (Eq. 11.40)

$$\psi(x) = A \left[ e^{ikx} - e^{i2\alpha - kx} \right].$$

so

$$e^{-2i\alpha} \left[ \frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)} \right] = -e^{2i\alpha}.$$ 

This is exact. For a very deep well, $E \ll V_0$, $k = \sqrt{2mE}/h \ll \sqrt{2m(E + V_0)/h} = k'$, so

$$e^{-2i\alpha} \left[ \frac{-ik' \cot(k'a)}{ik' \cot(k'a)} \right] = -e^{2i\alpha}; \quad e^{-2i\alpha} = e^{2i\alpha}; \quad \delta = -ka.$$
Problem 2: Probability Current of Scattered Wave

For the 3-D outgoing scattered wave far from the origin \((r \to \infty)\),

\[
\psi_{\text{scattered}} = A f(\theta, \phi) \frac{e^{ikr}}{r}
\]

show that the probability density current is

\[
j_{\text{scattered}} = \frac{\hbar k}{m r^2} |A|^2 |f|^2 \mathbf{u}_r
\]

where \(\mathbf{u}_r\) is a unit vector in the radial direction. Hint: Use the spherical coordinate form of the gradient operator to compute \(j\),

\[
\nabla = \frac{\partial}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{u}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \mathbf{u}_\phi
\]

where the \(\mathbf{u}'s\) are unit vectors in the respective coordinate directions.

In spherical coordinates the gradient is given by

\[
\nabla = \frac{\partial}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{u}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \mathbf{u}_\phi
\]

Then

\[
\nabla \psi_{\text{sc}} \xrightarrow{r \to \infty} Af \left( \frac{ik}{r} e^{ikr} - \frac{1}{r^2} e^{ikr} \right) \mathbf{u}_r + A \frac{\partial f}{\partial \theta} \frac{e^{ikr}}{r} \mathbf{u}_\theta + \frac{A}{r \sin \theta} \frac{\partial f}{\partial \phi} \frac{e^{ikr}}{r} \mathbf{u}_\phi
\]

Note that all except the first term vanish as \(1/r^2\) as \(r \to \infty\), while the first term goes as \(1/r\).

Thus retaining the first term only, we see that

\[
j_{\text{sc}} = \frac{\hbar}{2 \mu l} \left( \psi_{\text{sc}}^* \nabla \psi_{\text{sc}} - \psi_{\text{sc}} \nabla \psi_{\text{sc}}^* \right)
\]

\[
\xrightarrow{r \to \infty} \frac{\hbar}{2 \mu l} \mathbf{u}_r \left[ A^* f^* \frac{e^{-ikr}}{r} Af \frac{ik e^{ikr}}{r} - A^* f^* \frac{e^{ikr}}{r} A f (-ik) \frac{e^{-ikr}}{r} \right]
\]

\[
= \frac{\hbar k}{\mu r^2} |A|^2 |f|^2 \mathbf{u}_r,
\]
Problem 3: Partial Wave Analysis

A particle is scattered by a spherically symmetric potential at sufficiently low energy that the phase shifts $\delta_l = 0$ for all $l > 1$ (that is, only $\delta_0$ and $\delta_1$ are nonzero).

a. Show that the differential cross section has the form

$$\frac{d\sigma}{d\Omega} = A + B \cos \theta + C \cos^2 \theta$$

b. Determine $A$, $B$ and $C$ in terms of the phase shifts.

c. Determine the total cross section $\sigma$ in terms of the phase shifts.

\[ f(\theta) = \sum_l (2l+1) \frac{(e^{2i\delta_l} - 1)}{2ik} P_l(\cos \theta) = \frac{1}{k} \sum_l (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta) \]

For $\delta_0 \neq 0, \delta_1 \neq 0$, and all other $\delta_l$'s = 0,

\[ f(\theta) = \frac{1}{k} e^{i\delta_0} \sin \delta_0 + \frac{3}{k} e^{i\delta_1} \sin \delta_1 \cos \theta \]

and hence

\[ f^*(\theta)f(\theta) = \left(\frac{1}{k} e^{-i\delta_0} \sin \delta_0 + \frac{3}{k} e^{-i\delta_1} \sin \delta_1 \cos \theta\right) \left(\frac{1}{k} e^{i\delta_0} \sin \delta_0 + \frac{3}{k} e^{i\delta_1} \sin \delta_1 \cos \theta\right) \]

\[ = \frac{1}{k^2} \sin^2 \delta_0 + \frac{3}{k^2} \left(e^{i(\delta_0 - \delta_1)} + e^{-i(\delta_0 - \delta_1)}\right) \sin \delta_0 \sin \delta_1 \cos \theta + \frac{9}{k^2} \sin^2 \delta_1 \cos^2 \theta \]

\[ = A + B \cos \theta + C \cos^2 \theta \]

where

\[ A = \frac{1}{k^2} \sin^2 \delta_0 \quad B = \frac{6}{k^2} \cos(\delta_0 - \delta_1) \sin \delta_0 \sin \delta_1 \quad C = \frac{9}{k^2} \sin^2 \delta_1 \]

Thus

\[ \sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega |f|^2 = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta (A + B \cos \theta + C \cos^2 \theta) \]

\[ = 2\pi \left[ -A \cos \theta + \frac{B}{2} \sin^2 \theta - C \frac{1}{3} \cos^3 \theta \right]_{\theta=0}^{\theta=\pi} \]

\[ = 4\pi \left( A + \frac{C}{3} \right) \]
Problem 4: More Partial Wave Analysis

Consider a scattering situation in which only the \( l = 0 \) and \( l = 1 \) partial waves have appreciable phase shift.

a. Write an expression for the differential cross section \( d\sigma/d\Omega = |f(\theta)|^2 \) in terms of the phase shifts for the different partial waves. How does the contribution of the \( l = 1 \) partial wave affect the angular distribution of scattered particles, in comparison with just the \( l = 0 \) component?

b. What is the total cross section? Discuss how the contribution of the \( l = 1 \) wave affects this cross section.

c. What scattering measurement should be made to obtain an accurate estimate of the phase shift \( \delta_0 \) of the \( l = 0 \) component?

(a) Use the expansion of the differential cross section in terms of the partial waves with different \( l \), thus

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \left| \sum_{l=0}^{1} (2l + 1) P_l(\cos \theta)(e^{2i\delta_l} - 1) \right|^2
\]

(1)

where we have plugged in for the Legendre polynomials \( P_0 = 1 \) and \( P_1 = \cos \theta \).

For \( l = 0 \) there is no angular dependence (this is “S-wave” scattering). The \( l = 1 \) wave contribution adds an angular dependence that favors the forward direction (close to \( \theta = 0 \)).

(b) The total cross section is

\[
\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{1} (2l + 1) \sin^2 \delta_l
\]

(3)

\[
= \frac{4\pi}{k^2} (\sin^2 \delta_0 + 3 \sin^2 \delta_1)
\]

(4)

thus the \( l = 1 \) contribution increases the cross section.

(c) Measure the differential cross-section at an angle where the \( l = 1 \) component has no contribution. This happens when \( \theta = \pi/2 \), so that \( \cos \theta = 0 \) and

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \left| (e^{2i\delta_0} - 1) \right|^2 = \frac{1}{k^2} \sin^2 \delta_0
\]
Problem 5: Born Approximation

Use the Born approximation to compute the total scattering cross section $\sigma$ for particles of mass $m$ from an attractive Gaussian potential,

$$V(r) = -V_0 e^{-(r/a)^2}$$

The scattering amplitude $f(\theta, \phi)$ is proportional to the 3D Fourier transform of the potential (recall that the FT of a Gaussian is another Gaussian – you can look this up, but be careful to keep track of all the coefficients),

$$f(\theta, \phi) = -\frac{m}{2\pi \hbar^2} \int e^{-iq'r} V(r') d^3r'$$

$$= \frac{mV_0}{2\pi \hbar^2} \int e^{-iq'r} e^{-r^2/a^2} d^3r'$$

$$= \frac{\sqrt{\pi} m V_0 a^3}{2\hbar^2} e^{-q^2a^2/4}$$

(5)  (6)  (7)

The differential cross section is then

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\pi m^2 V_0^2 a^6}{4\hbar^4} e^{-q^2a^2/2}$$

(8)

The total cross section is the integral of this over all solid angle. Before doing this integral, recall that

$$q^2 = 4k^2 \sin^2(\theta/2)$$

Now,

$$\sigma = \int \left( \frac{d\sigma}{d\Omega} \right) d\Omega$$

$$= \frac{\pi m^2 V_0^2 a^6}{4\hbar^4} \int_0^{2\pi} d\phi \int_0^\pi e^{-2k^2a^2 \sin^2(\theta/2)} \sin \theta d\theta$$

(9)

The integral over $\phi$ just multiplies the whole integral by $2\pi$. Substitute $x = 2k^2a^2 \sin^2(\theta/2)$, thus $dx = k^2a^2 \sin \theta d\theta$ so that

$$\sigma = \frac{\pi^2 m^2 V_0^2 a^6}{2\hbar^4} \int_0^{2k^2a^2} \frac{1}{k^2a^2} e^{-x} dx$$

$$= \frac{\pi^2 m^2 V_0^2 a^4}{2\hbar^4 k^2} \left( 1 - e^{-2k^2a^2} \right)$$

(10)  (11)