1. Robertson-Walker metric
Show that the Robertson-Walker metric
\[ c^2d\tau^2 = c^2dt^2 - R^2(t)[dr^2 + S_k^2(r)d\psi^2] \]
can also be written in the form
\[ c^2d\tau^2 = c^2dt^2 - R^2(t)[dr^2/(1 - kr^2) + r^2d\psi^2] \]

2. Metric for an Open Universe
For a \( k = -1 \) Friedmann cosmology (\( \Lambda = 0 \)), with \( \rho = p = 0 \), show that the R-W metric line element becomes
\[ c^2d\tau^2 = c^2dt^2 - c^2t^2[dr^2 + \sinh^2 r d\psi^2] \]

3. Relativistic Velocities
(a) Show that the general relativistic relation between recession velocity and cosmological redshift is
\[ v_{\text{rec}}(t, z) = \frac{c}{R_0} \frac{\dot{R}(t)}{H(z')} \int_0^z \frac{dz'}{H(z')} \]
where \( H(z') \) is the Hubble constant at redshift \( z' \).
(b) Show that the special relativistic relation between peculiar velocity and redshift is
\[ v_{\text{pec}}(z) = c\left(1 + \frac{1}{z}ight)^2 - 1 \]
(c) Show that both relativistic relations are approximately \( v \approx cz \) at small distance.

4. Particle Horizon
Following the suggestions outlined on pp. 85-86 of the text, show that the dominant form of mass-energy at early times must scale as \( \rho \propto R^{-\alpha} \) with \( \alpha > 2 \) for a particle horizon to exist.

5. Evolution of \( \Omega_{\text{matter}}, \Omega_{\text{vac}} \)
Compute how the density parameters \( \Omega_{\text{matter}} \) and \( \Omega_{\text{vac}} \) evolve with time in different cosmologies and plot the results on a figure like 3.5 on p. 83. In other words, for each choice of \( \Omega_{\text{matter}}, \Omega_{\text{vac}} \) today, plot the trajectory of the model backwards in time to where it would lie in that same diagram at \( t \approx 0 \). On the plot, clearly indicate which end of each curve is now and which is at \( t = 0 \). Pick models from all regions of the diagram and be sure to include the following (in other words, do more than just these):
\[ \Omega_{\text{matter}} = 1, \Omega_{\text{vac}} = 0 \]
\[ \Omega_{\text{matter}} = 0.2, \Omega_{\text{vac}} = 0 \]
\[ \Omega_{\text{matter}} = 0.3, \Omega_{\text{vac}} = 0.7 \]