Nonlinear Dynamics

PHYS 750

Problem Set: Optional
Distributed February 24, 2005
Not Due Whenever

1. The return map that you obtain from the Rössler attractor with canonical control parameter values is part of a parabola. It looks like the logistic map
\[ x' = \lambda x (1 - x) \]
except that the left-hand edge of the parabola does not go all the way down to zero.

The actual form of the return map can be approximated by adding a linearly decreasing term to the logistic map:
\[ x' = f(x; \lambda, \alpha) = \lambda x (1 - x) + \alpha (1 - x) \]

These terms can be combined to the functional form
\[ f(x; \lambda, \alpha) = (\lambda x + \alpha)(1 - x) \]

This family of maps does not satisfy the conditions for unimodal maps of the interval, since one of the conditions is that \( f(0) = f(1) \). Therefore the intuition gained from study of the logistic map may be wrecked.

a. Show that this function has a maximum at
\[ x = \frac{\lambda - \alpha}{2\lambda} \]

b. Show that the value of \( f \) at the maximum is \( (\lambda + \alpha)^2 / 4\lambda \). We require that \( f(x_{\text{max}}) = 1 \). Why?

c. What constraint does this condition put on the two parameters \( \lambda \) and \( \alpha \)? Solve for \( \lambda \) as a function of \( \alpha \) and show
\[ \lambda = (2 - \alpha) + 2\sqrt{1 - \alpha} \]

d. When \( \alpha = 0 \) every possible trajectory (sequence of 0’s and 1’s) can be found. In particular there is a full spectrum of unstable periodic orbits. As \( \alpha \) increases from 0 some of these orbits must be destroyed. The earliest orbits that are destroyed have long sequences of 0000 .. 00 in them. Why? (Hint: look at
the cobweb diagram for this return map, and start an orbit off at \( x = 0 \). How many iterates can it have before it reaches the maximum?

e. Identify values \( \alpha_n \), below which orbits with \( n \) successive 0's can exist, and above which they cannot. This is one version of the "pruning" idea: orbits are pruned away as some control parameter is varied.