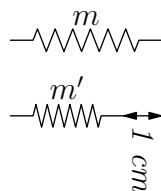


Homework 5

Chapter 9

Problem 1. By what fraction does the mass of a $m = 10 \text{ g}$, $k = 500 \text{ N/m}$ spring increase when it is compressed by 1 cm ?



Compression increases the potential energy of the spring by

$$\Delta U = \frac{1}{2}k\Delta x^2 = \frac{1}{2} \cdot 500 \text{ N/m} \cdot (0.01 \text{ m})^2 = 25.0 \text{ mJ} . \quad (1)$$

From Einstein's mass-energy equivalence, increasing the spring's energy must also increase its mass, since mass and energy are two ways of talking about the same stuff.

$$\Delta E = \Delta mc^2 \quad (2)$$

$$\Delta m = \frac{\Delta E}{c^2} = \frac{\Delta U}{c^2} = \frac{0.025 \text{ J}}{(3 \cdot 10^8 \text{ m/s})^2} = 2.78 \cdot 10^{-19} \text{ kg} . \quad (3)$$

So the fractional mass increase is

$$\frac{\Delta m}{m} = \frac{2.78 \cdot 10^{-19} \text{ kg}}{0.010 \text{ kg}} = 2.78 \cdot 10^{-17} . \quad (4)$$

This mass difference is quite small, which is why it took so long to come up with the $E = mc^2$ idea. Notice, though, that the mass difference is equal to the mass of 16 billion protons (at $1.67 \cdot 10^{-27} \text{ kg}$ a pop). Nuclear reactions achieve their high energies through small energy changes for an *enormous* number of nuclei (on the order of Avogadro's number $N_A = 6.022 \cdot 10^{23}$ particles/mole)

Problem 2. Returning to Earth in his rocket ship at $0.6c$, Jack calls Jill (who is on Earth) to let her know he's almost home. He sets up his radio transmitter to broadcast at 500 MHz . (a) In Jill's frame, how much time passes between two successive maxima of the voltage across Jack's antenna? (b) What is the frequency of the signal as it reaches Jill?

(a) Jack's carrier wave has a period of

$$T = \frac{1}{f} = 2.00 \text{ ns} . \quad (5)$$

Jill thinks Jack's clock is time-dilated (i.e. runs slower), so the time between voltage maxima in Jill's frame is

$$\Delta t' = \gamma T = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} T = 1.25 \cdot T = 2.50 \text{ ns} . \quad (6)$$

Note that this time between antenna voltage maxima is *not* the period of the light Jill receives, because Jack is moving towards her (this would be the period she received if Jack were moving perpendicular to her).

(b) During the time $\Delta t'$ between successive voltage maxima, Jack came $v\Delta t'$ closer to Earth while the earlier maximum came $c\Delta t'$ closer. The wavelength of the signal Jill receives is thus

$$\lambda' = (c - v)\Delta t' = 0.4c\Delta t' = 0.300 \text{ m} , \quad (7)$$

and the frequency is

$$cT' = \lambda' \quad (8)$$

$$f' = \frac{1}{T'} = \frac{c}{\lambda'} = 1.00 \text{ GHz} . \quad (9)$$

If we wanted to leave the whole problem in symbolic notation, we have

$$f' = \frac{c}{\lambda'} = \frac{c}{(c - v)\Delta t'} = \frac{1}{(1 - \frac{v}{c})\gamma T} = \frac{\sqrt{1 - (\frac{v}{c})^2}}{1 - \frac{v}{c}} f = \frac{\sqrt{(1 + \frac{v}{c})(1 - \frac{v}{c})}}{1 - \frac{v}{c}} f = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} f , \quad (10)$$

which is the source of the text's Equation 24.22.

Problem 3. (a) How long (in Earth time) does it take for a plane to circle the Earth at low altitude (average radius 6,370 km) going 600 mph? (b) How much less time passes according to a clock on the plane?

For the purpose of this problem, you may assume the Earth is stationary and not rotating.

(a)

$$600 \text{ mph} \cdot \frac{1.6 \text{ km}}{1 \text{ mile}} \cdot \frac{1 \text{ hour}}{3600 \text{ s}} = 267 \text{ m/s} \quad (11)$$

$$v\Delta t = \Delta x = 2\pi r \quad (12)$$

$$\Delta t = \frac{2\pi r}{v} = \frac{2\pi \cdot 6.37 \cdot 10^6 \text{ m}}{267 \text{ m/s}} = 150 \text{ ks} = 41.7 \text{ hours} \quad (13)$$

(b) This is pretty much a twin-paradox, with the clock on the constantly-accelerating plane running slower than the clock on Earth.

$$\Delta t' = \frac{\Delta t}{\gamma} = \sqrt{1 - \left(\frac{v}{c}\right)^2} \Delta t = (1 - 3.95 \cdot 10^{-13}) \Delta t \quad (14)$$

$$\Delta t - \Delta t' = 3.95 \cdot 10^{-13} \Delta t = 59.3 \text{ ns} \quad (15)$$

Problem 4. You are working in the radiology department at a veterinary hospital. The cat you are trying to X-ray refuses to hold still so you volunteer to stand at the table and hold the cat down during the exposure. If X-rays of wavelength $\lambda = 24 \text{ pm}$ enter the cat from above, what is the wavelength of the photons that enter your head after scattering through a 135° angle?

Compton effect

$$\lambda' - \lambda_0 = \frac{h}{m_e c} (1 - \cos \theta) \quad (16)$$

$$\lambda' = \lambda_0 + \frac{h}{m_e c} (1 - \cos \theta) = \lambda_0 + \frac{hc}{m_e c^2} (1 - \cos \theta) = 24 \text{ pm} + \frac{1240 \text{ eV}\cdot\text{nm}}{511 \text{ keV}} (1 - \cos 135^\circ) = 28.1 \text{ pm} . \quad (17)$$

These are still quite penetrative X-rays, which is why radiologists try to hold down cats with tape and sandbags when taking radiographs. Determining the intensity of the scattered beam is more complicated and would involve some sort of volume integral over the atoms in the cat and the scattering cross section per atom.

Problem 5. The resolving power of a microscope depends on the wavelength of light used. If you want to “see” an atom, you must resolve features on the order of 0.1 \AA . (a) If you use electrons (in an electron microscope), what minimum kinetic energy would they require? (b) If you used photons (in a light microscope), what minimum kinetic energy would they require?

(a) This is a de Broglie problem, since the resolution of the microscope is on the order of the wavelength of the probe photon or electron. We use de Broglie’s formula to relate the particle’s wavelength to its momentum $\lambda p = h$ (Equation 28.10).

First, assume our electrons have non-relativistic speeds and we can use $p = m_0 v$ (as opposed to the relativistic Equation 9.15 $p = \gamma m_0 v$).

$$\lambda = \frac{h}{p} = \frac{h}{m_0 v} \quad (18)$$

$$v = \frac{h}{m_0 \lambda} \quad (19)$$

$$K = \frac{1}{2} m_0 v^2 = \frac{h^2}{2 m_0 \lambda^2} . \quad (20)$$

We can resolve features on the order of a wavelength, so let’s set $\lambda = 0.1 \text{ \AA}$.

$$K = \frac{6.626 \cdot 10^{-34} \text{ J}\cdot\text{s}^2}{2 \cdot 9.11 \cdot 10^{-31} \text{ kg} \cdot (1 \cdot 10^{-11} \text{ m})^2} = 2.41 \cdot 10^{-15} \text{ J} = 15.1 \text{ keV} . \quad (21)$$

We assumed that the electrons were non-relativistic, so we check our calculated speed

$$v = \frac{h}{m_0 \lambda} = 0.727 \cdot 10^8 \text{ m/s} = 0.242c . \quad (22)$$

This is on the border of the relativistic behavior, so we can go back and redo the calculation relativistically.

$$p = \frac{h}{\lambda} = 6.63 \cdot 10^{-23} \text{ kg}\cdot\text{m/s}^2 \quad (23)$$

$$E^2 = p^2 c^2 + m_0^2 c^4 \quad (24)$$

$$K = E - E_{\text{rest}} = E - m_0 c^2 = \sqrt{p^2 c^2 + m_0^2 c^4} - m_0 c^2 = 2.38 \cdot 10^{-15} \text{ J} = 14.8 \text{ keV} . \quad (25)$$

Which is pretty close to our non-relativistic answer. The E^2 formula is simply a rephrasing of $E = \gamma m_0 c^2$ in terms of momentum. The derivation is sketched out in the text around Equation 9.22.

If you wanted to get really fancy, you could use the formula for the resolution of a circular-aperture microscope (Equation 27.15)

$$\theta_{\min} = 1.22 \frac{\lambda}{D} \quad (26)$$

$$\Delta x_{\min} = L \tan(\theta_{\min}) \quad (27)$$

to determine the required wavelength of light, but you'd have to make guesses about the diameter of the aperture D and the distance between the aperture and the specimen L .

(b) For light, we use the relativistic formula with $m_0 = 0$

$$E = K = pc = \frac{hc}{\lambda} = \frac{1240 \text{ eV}\cdot\text{nm}}{0.01 \text{ nm}} = 124 \text{ keV} = 1.98 \cdot 10^{-14} \text{ J}, \quad (28)$$

around 8 times larger than the energy needed using electrons.

Problem 6. (a) How fast would you have to be driving a 20 ft long limo to fit into 15 ft deep garage? (b) How deep would the garage appear to the driver of the limo? Is the limo ever really entirely in the garage? Explain any apparent paradoxes. Both of the lengths given are proper lengths.

(a) The proper length of the limo L_0 needs to be contracted to the length of the garage (equations compressed for space, read right to left, then top to bottom).

$$L = \frac{L_0}{\gamma} \quad \frac{L}{L_0} = \frac{1}{\gamma} = \sqrt{1 - \left(\frac{v}{c}\right)^2} \quad (29)$$

$$\left(\frac{L}{L_0}\right)^2 = 1 - \left(\frac{v}{c}\right)^2 \quad \left(\frac{v}{c}\right)^2 = 1 - \left(\frac{L}{L_0}\right)^2 \quad (30)$$

$$\frac{v}{c} = \sqrt{1 - \left(\frac{L}{L_0}\right)^2} = 0.661 \quad v = 0.661c = 198 \cdot 10^6 \text{ m/s}. \quad (31)$$

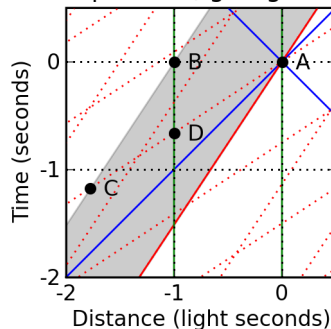
(b) The garage L_{g0} is length contracted in the limo frame

$$L_g = \frac{L_{g0}}{\gamma} = L_{g0} \sqrt{1 - \left(\frac{v}{c}\right)^2} = 15 \text{ ft} \sqrt{1 - .661^2} = 11.2 \text{ ft} = 3.43 \text{ m}. \quad (32)$$

Wait, how can the limo fit into a garage that appears even shorter than its proper length of 15 ft? This is a relative-simultaneity effect like the muon clock running slower than an earth clock while the earth clock runs slower than the muon clock. In the garage frame, the limo-nose-passes-crashes-into-back-wall event A and the limo-tail-passes-door event B occur at the same time. In the limo frame, event A happens some time before event B .

Drawing a space-time diagram in the garage frame may help clarify the different events. The limo is the grey smear. The red dotted lines represent a 1 ls time and space grid for the limo driver. The blue lines show the speed of light. The green lines show the garage. I've rescaled the garage and limo to make them 1 ls and 1.33 ls long respectively so that the axes have simple labels. The limo driver thinks that at the same time as event A , the tail of the limo is back at event C , while the door of the garage is up at event D . Note that the garage is less than 1 ls deep in the limo frame.

Limo paradox, garage frame



“Entirely in the garage” is something of a trick question, since it means “all of the limo is in the garage at same time” and “at the same time” depends on your reference frame. In the garage frame, the limo is entirely in the garage for a single instant. In the limo frame, the limo is never entirely in the garage. An absolute, true-no-matter-what answer to the question is not possible in the relativistic world-view.