Homework 2

Chapters 14 and 24

Problem 1. *A string with a mass of* 5 *g and a length of* 1 *m has one end attached to a wall.* 70 *cm from the wall, the string passes over a pully and hangs, supporting a* 1 *kg mass. (a) What is the fundamental frequency of vibration? (b) What is the frequency of second harmonic?*

(a) The linear density of the string is

$$\mu = \frac{m}{L} = \frac{5\text{ g}}{1\text{ m}} = 5\text{ g/m} \ . \tag{1}$$

The tension of the string is

$$T = Mg = 1\text{ kg} \cdot 9.8\text{ m/s}^2 = 9.8\text{ N} \ . \tag{2}$$

The speed of wave-propagation in the string is

$$v = \sqrt{\frac{T}{\mu}} = 44.3\text{ m/s} \ . \tag{3}$$

For the fundamental frequency, the distance between the fixed ends is half a wavelength, so a moving wave crosses it in half a period.

$$\Delta x = v\Delta t = v \cdot \frac{1}{2f_1} \tag{4}$$

$$f_1 = \frac{v}{2\Delta x} = \frac{44.3\text{ m/s}}{2 \cdot 0.70\text{ m}} = 31.6\text{ Hz} \tag{5}$$

(b) For the second harmonic, the distance between the fixed ends is a full wavelength, so a moving wave crosses it in a full period.

$$\Delta x = v \cdot \frac{1}{f_2} \tag{6}$$

$$f_2 = 2f_1 = 63.2\text{ Hz} \tag{7}$$

Problem 2. *A student uses an audio oscillator of adjustable frequency to measure the depth of the well. The student hears two successive resonances at* 122.0 *Hz and* 127.1 *Hz. How deep is the well?*

We can approximate the well as a cylinder closed at the bottom and open at the top. As a result, the well resonates at (Equation 14.11)

$$f_n = \frac{nv}{4L} \ . \tag{8}$$

It's 5°C outside my window at the moment, so the speed of sound is around (Equation 13.27)

$$v = 311\text{ m/s} + (0.6\text{ m/s}\cdot{}^\circ\text{C})T = 314\text{ m/s} \ . \tag{9}$$

Let the resonance at $f_L = 122$ Hz be the n^{th} harmonic. Then the $f_H = 127.1$ Hz harmonic is the $(n+2)^{\text{th}}$ harmonic (as the next successive resonance in a pipe closed at one end). Plugging these into Equation 8 we have

$$\frac{f_H}{f_L} = \frac{\frac{(n+2)v}{4L}}{\frac{nv}{4L}} = \frac{n+2}{n} = 1 + \frac{2}{n} \tag{10}$$

$$\frac{2}{n} = \frac{f_H}{f_L} - 1 \tag{11}$$

$$n = \frac{2}{\frac{f_H}{f_L} - 1} = \frac{2f_L}{f_H - f_L} = 48 \ . \tag{12}$$

Hmm, 48 is not an odd harmonic, and all harmonics of pipes open at one end and closed at the other are odd. This means that *your TA made a mistake* writing the homework problem, at which point you should start emailing questions and complaints. (corrected problem) If you want two numbers that work, you can use 126 Hz and 138 Hz, in which case

$$n = \frac{2f_L}{f_H - f_L} = 21 \ . \tag{13}$$

Now that we know the f_L is the 21$^{\text{st}}$ harmonic, we can plug into Equation 8 and find the well depth

$$f_L = \frac{nv}{4L} \tag{14}$$

$$L = \frac{nv}{4f_L} = \frac{21 \cdot 314 \text{ m/s}}{4 \cdot 126 \text{ Hz}} = 13.1 \text{ m} \tag{15}$$

(open well) On the other hand, perhaps the reason for the even result for n is that the well is not a well after all, but a pipe passing down into a vertical cystern (`http://en.wikipedia.org/wiki/Cistern`) where there is a layer of air before the level of the water. In that case, the pipe is open to air at both ends and even harmonics are allowed, so

$$f_n = \frac{nv}{2L} \tag{16}$$

$$\frac{f_H}{f_L} = \frac{\frac{(n+1)v}{2L}}{\frac{nv}{2L}} = \frac{n+1}{n} = 1 + \frac{1}{n} \tag{17}$$

$$\frac{1}{n} = \frac{f_H}{f_L} - 1 \tag{18}$$

$$n = \frac{1}{\frac{f_H}{f_L} - 1} = \frac{f_L}{f_H - f_L} = 24 \tag{19}$$

$$L = \frac{nv}{2f_L} = \frac{24 \cdot 314 \text{ m/s}}{2 \cdot 122 \text{ Hz}} = 30.9 \text{ m} \tag{20}$$

Apologies for the mistake.

Problem 3. *Set-top loop antennas are sometimes used to pick up UHF TV broadcast with a carrier frequency f and peak electric field at the antenna of E_{max}. The changing magnetic flux in the antenna loop produces an emf which matches the broadcast signal. (a) Using Faraday's law, derive an expression for the amplitude of the emf in a single-turn circular loop of radius r, if r is much less than the broadcast wavelength. (b) If the TV station is due East of your house and the electric field oscillates vertically, how would you orient the antenna for best reception?*

(a) Faraday's law is given in Equation 24.6

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \,. \tag{21}$$

The left-hand side of Faraday's law is the induced emf. Working on the right-hand side and noting that $\Phi_B = \mathbf{A} \cdot \mathbf{B} = AB\cos(\theta)$, we see

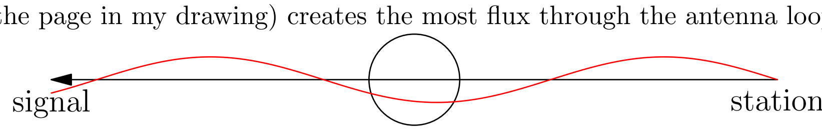
$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -A\cos(\theta)\frac{\mathrm{d}B}{\mathrm{d}t} = -\pi r^2 \cos(\theta)\frac{\mathrm{d}B}{\mathrm{d}t} = -\pi r^2 \cos(\theta)\frac{\mathrm{d}}{\mathrm{d}t}\left(B_{\max}\sin(kx - wt)\right) = \pi r^2 \cos(\theta)\omega B_{\max}\cos(kx - wt) \,. \tag{22}$$

So the amplitude of the emf is

$$A_{\text{emp}} = \pi r^2 \omega B_{\max}\cos(\theta) = 2\pi^2 r^2 f B_{\max}\cos(\theta) = 2\pi^2 r^2 f \frac{E_{\max}}{c}\cos(\theta) \,. \tag{23}$$

(b)

It is best to have the antennal oriented in the plane of the electric field (red), so that the magnetic field (perpendicular to the page in my drawing) creates the most flux through the antenna loop.



Problem 4. *(a) Caluculate the inductance of an LC circuit that oscillates at 60 Hz when the capacitance is 5.00 μF. (b) A resistor is inserted into the LC loop shown in Figure 24.8. Give a qualitative description of current oscillation in the new circuit.*

(a) The oscillation frequency of an LC circuit is given in Equation 24.24 (with the derivation in the preceeding few equations)

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \ . \tag{24}$$

So

$$LC = \frac{1}{(2\pi f_0)^2} \tag{25}$$

$$L = \frac{1}{C(2\pi f_0)^2} = 1.41 \text{ H} \tag{26}$$

(b) When current passes through the resistor some electrical energy is converted into heat, so the LRC circuit would act as a damped harmonic oscillator (see Section 12.6 for more on damped oscillations).

Problem 5. *You're listening to WKDU (transmitted from Van Rensselaer Hall) on 91.7fm while watching a basketball game 400 m away at the DAC. How many wavelengths are between you and the transmitter? FM channel names give the carrier frequency in MHz.*

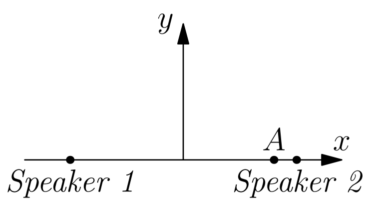
The wavelength of the signal is

$$\lambda = \Delta x = v\Delta t = cT = \frac{c}{f} = 3.27 \text{ m} \tag{27}$$

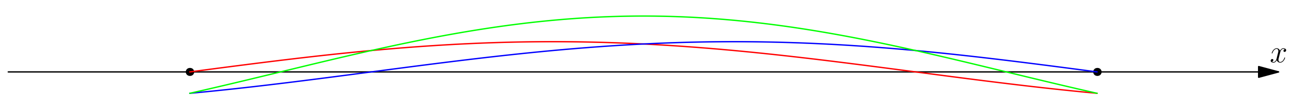
So the number of wavelengths in 400 m is

$$N = \frac{L}{\lambda} = \frac{400}{3.27} = 122 \tag{28}$$

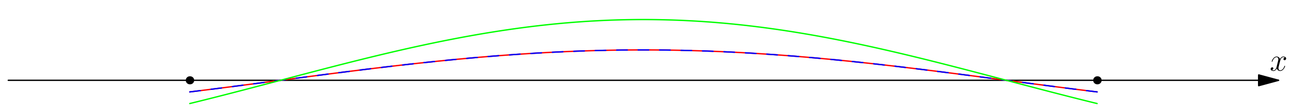
Problem 6. BONUS PROBLEM. *Two identical speakers $d = 10.0$ m apart are driven by the same oscillator with a frequency of $f = 21.5$ Hz. (a) Explain why a reciever at point A records a minimum in sound intensity from the two speakers. (b) If the reciever is moved in the plane of the speakers, what path should it take so that the intensity remains at a minimum? That is, determine the relationship between x and y (the coordinates of the reciever) that causes the receiver to record a minimum in sound intensity. Take the speed of sound to be $v = 344$ m/s.*



(a) To get a feeling for why a minimum exists, consider the intensity of output sound when the amplitude of the wave generated by a speaker is zero at the generating speaker (the wavelength is given by $\lambda = v/f = 16.0$ m).



Note the existence of two nodes in the sum (green). This snapshot occurs just before the standing wave reaches it's maximum amplitude, because a small time later the wave from the left speaker (red) will have moved to the right, the wave from the left speaker (blue) will have moved to the left, and the peaks will both arive in the middle, forming



The easiest wat to think about this is to focus on the phase difference between the two waves as a function of position. The two speakers are broadcasting in phase, but by the time the second wave reaches the first speaker it is $2\pi \cdot d/\lambda = 3.93$ rad out of phase. The nodes come when the sound from the two speakers are out of phase by π rad $= 180°$. Letting $x = \pm d/2$ be the position of the two speakers, we have

$$\phi_L = 2\pi\frac{x + d/2}{\lambda} \mod 2\pi \tag{29}$$

$$\phi_R = -2\pi\frac{x + d/2}{\lambda} \mod 2\pi \tag{30}$$

$$\Delta\phi = \phi_L - \phi_R = \frac{4\pi x}{\lambda} \mod 2\pi \ , \tag{31}$$

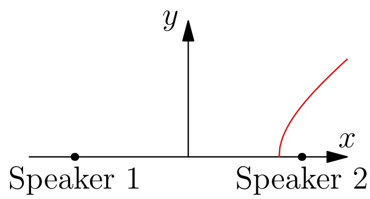
so the sound is in phase at $x = 0$ (as you'd expect), and we get our peak amplitude there. The nodes occur at

$$\pi = \Delta\phi(x_n) = \frac{4\pi x_n}{\lambda} \mod 2\pi \tag{32}$$

$$x_n = \pm\frac{\lambda}{4} = \pm 4.00 \text{ m} \tag{33}$$

(b) Extending this reasoning into the plane of the speakers, we know beforehand that nodes will lie along hyperbola, because hyperbola are "the locus of points where the difference of the distances to the two foci is a constant". If $\lambda/4$ is the distance from the origin to the right hand node on the line between the two foci (which we found in (a)), and $d/2$ is the distance from the origin to either focus, then the appropriate hyperbola is

$$1 = \frac{x^2}{\lambda^2/16} - \frac{y^2}{d^2/4 - \lambda^2/16} = \frac{x^2}{16 \text{ m}^2} - \frac{y^2}{25 \text{ m}^2 - 16 \text{ m}^2} = \frac{x^2}{16 \text{ m}^2} - \frac{y^2}{9 \text{ m}^2} . \tag{34}$$



If you don't remember that much about hyperbolas, you can grind through the algebra and get the same result.

$$\phi_L = 2\pi\frac{\sqrt{(x + d/2)^2 + y^2}}{\lambda} \mod 2\pi \tag{35}$$

$$\phi_R = 2\pi\frac{\sqrt{(x - d/2)^2 + y^2}}{\lambda} \mod 2\pi \tag{36}$$

$$\Delta\phi = \phi_L - \phi_R = \frac{2\pi}{\lambda}\left[\sqrt{(x + d/2)^2 + y^2} - \sqrt{(x - d/2)^2 + y^2}\right] \mod 2\pi . \tag{37}$$

If we decide to follow only the node on the right (where $\Delta\phi = \pi$), we can dispense with the mod 2π yielding

$$\pi = \Delta\phi = \frac{2\pi}{\lambda}\left[\sqrt{(x + d/2)^2 + y^2} - \sqrt{(x - d/2)^2 + y^2}\right] \tag{38}$$

$$\frac{\lambda}{2} = \sqrt{(x + d/2)^2 + y^2} - \sqrt{(x - d/2)^2 + y^2} . \tag{39}$$

Focusing on the right hand side of the equation, we see

$$\sqrt{(x \pm d/2)^2 + y^2} = \sqrt{x^2 \pm dx + d^2/4 + y^2} = \sqrt{a \pm b} , \tag{40}$$

with

$$a \equiv x^2 + \frac{d^2}{4} + y^2 \qquad\qquad b \equiv dx . \tag{41}$$

Going back to our main equation in terms of a and b

$$\frac{\lambda}{2} = \sqrt{a + b} - \sqrt{a - b} \tag{42}$$

$$\frac{\lambda^2}{4} = \left(\sqrt{a + b} - \sqrt{a - b}\right)^2 \tag{43}$$

$$= \sqrt{(a + b)^2} + \sqrt{(a - b)^2} - 2\sqrt{(a + b)(a - b)} \tag{44}$$

$$= (a + b) + (a - b) - 2\sqrt{a^2 - b^2} \tag{45}$$

$$= 2a - 2\sqrt{a^2 - b^2} \tag{46}$$

$$\frac{\lambda^2}{4} - 2a = -2\sqrt{a^2 - b^2} \tag{47}$$

$$\frac{\lambda^4}{16} - \lambda^2 a + 4a^2 = 4(a^2 - b^2) \tag{48}$$

$$\frac{\lambda^4}{16} - \lambda^2 a + 4b^2 = 0 . \tag{49}$$

Now that we've gotten rid of the square roots, we go back and plug in for a and b.

$$\frac{\lambda^4}{16} - \lambda^2 x^2 - \frac{\lambda^2 d^2}{4} - \lambda^2 y^2 + 4d^2 x^2 = 0 \tag{50}$$

$$\frac{\lambda^2}{16}\left(\lambda^2 - 4d^2\right) - \lambda^2 y^2 + (4d^2 - \lambda^2)x^2 = 0 \tag{51}$$

$$(4d^2 - \lambda^2)x^2 - \lambda^2 y^2 = \frac{\lambda^2}{16}\left(4d^2 - \lambda^2\right) \tag{52}$$

$$x^2 - \frac{\lambda^2 y^2}{4d^2 - \lambda^2} = \frac{\lambda^2}{16} \tag{53}$$

$$\frac{16x^2}{\lambda^2} - \frac{16y^2}{4d^2 - \lambda^2} = 1 \tag{54}$$

$$\frac{x^2}{\lambda^2/16} - \frac{y^2}{d^2/4 - \lambda^2/16} = 1\ , \tag{55}$$

which is the same equation our knowledge of the hyperbola gave us earlier.