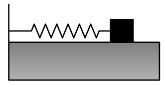
Homework 1

Chapters 12 and 13

Problem 1. *A frictionless block-spring system oscillates with amplitude A. If the mass of the block is doubled without changing the amplitude, (a) does the total energy change? (b) does the frequency of oscillation change?*

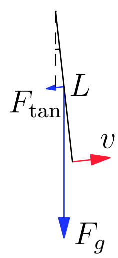


(a) The total energy is equal to the spring potential energy at maximum extension (when the kinetic energy is zero), so $E_{\mathrm{T}} = \frac{1}{2}kA^2$. Neither the spring constant, nor the amplitude changed, so the total energy is unaffected.
(b) The heavier mass will move more slowly under the influnce of the same spring, so the frequency is smaller for the bigger mass. Quantitatively

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \; , \tag{1}$$

so doubling the mass reduces the frequency to $f' = \frac{f}{\sqrt{2}}$.

Problem 2. *A thin, rigid rod $L = 8.4$ m long pivots freely about one end. The rod is initially deflected $\theta_i = 6.4°$ from the vertical with an angular velocity of $\dot{\theta}_i = 2.7°/s$. (a) Determine the time dependence $\theta(t)$. (b) By what angle is the rod deflected at $t = 8.9$ s?*
Hint: you might want to review torque and moments of inertia in Chapter 10.



The only force on the rod is from gravity, with mg pulling the rods center of mass downward. Only the portion of this force that is perpendicular to the rod itself (tangential to the circle the rod sweeps out) affects its rotation. The torque on the rod is thus

$$\tau = -F_{\mathrm{tan}} \cdot \frac{L}{2} = -F_g \sin(\theta) \cdot \frac{L}{2} = -\frac{mgL}{2}\sin(\theta) \approx -\frac{mgL}{2}\theta \; , \tag{2}$$

where we used the small angle approximation $\sin(\theta) \approx \theta$ for the last step.
The equation of motion is then

$$\tau = I\ddot{\theta} = \frac{1}{3}mL^2\ddot{\theta} \; , \tag{3}$$

because the moment of inertia of a rod rotating about it's end is $I = \frac{1}{3}mL^2$ (Table 10.2).
Combining the two expressions of τ we have

$$-\frac{mgL}{2}\theta = \frac{1}{3}mL^2\ddot{\theta} \tag{4}$$

$$-\frac{3g}{2L}\theta = \ddot{\theta} \; . \tag{5}$$

Comparing this formula to Equation 12.5 for a general simple harmonic oscillator

$$\ddot{x} = -\omega^2 x \; , \tag{6}$$

we see by matching that

$$\omega = \sqrt{\frac{3g}{2L}} \approx 1.323 \text{ rad/s} \; . \tag{7}$$

We can plug this ω into Equation 12.6

$$\theta(t) = A\cos(\omega t + \phi) \; , \tag{8}$$

where A and ψ are determined by the initial conditions (see Example 12.3)

$$\theta_i = 6.4°/\text{s} \cdot \frac{\pi}{180°} \approx 111.7 \text{ mrad} \tag{9}$$

$$\dot{\theta}_i = 2.7°/\text{s} \cdot \frac{\pi}{180°} \approx 47.12 \text{ mrad/s} \tag{10}$$

$$\phi = \arctan\left(\frac{-\dot{\theta}_i}{\omega\theta_i}\right) \approx -308.7 \text{ mrad} \approx -17.69° \tag{11}$$

$$A = \sqrt{\theta_i^2 + \left(\frac{\dot{\theta}_i}{\omega}\right)^2} \approx 117.2 \text{ mrad} \approx 6.718° \tag{12}$$

$$\theta(t) \approx 0.1172\cos(1.323t - 0.3087) . \tag{13}$$

(b) Plugging in $t = 8.9$ s yields

$$\theta(t = 8.9 \text{ s}) = 0.1172\cos(1.323 \cdot 8.9 - 0.3087) \approx 53 \text{ mrad} \approx 3.0° . \tag{14}$$

Problem 3. *Hydraulic shock absorbers typically consist of a piston in an oil filled reservoir (see* `http://en.wikipedia.org/wiki/Shock_absorber`*). Orifices in the piston allow oil to flow from one side of the piston to the other, so piston movement stirs the oil. The stirring transforms the piston's mechanical and kinetic energy into heat, damping any piston oscillation.*
You are asked to design a shock absorber for a motor-unicycle (suspended by a single shock and spring). With a rider the motor-unicyle weighs $m = 140$ kg and is sprung with a $k = 2.0$ kN/m spring.
(a) Determine the undamped resonant frequency in Hz. (b) Determine the damping coefficient for critical damping.
Hint: treat the shock absorber as a damped simple harmonic oscillator.

(a) The frequency of underdamped vibration is given by

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{15}$$

Without damping ($b = 0$),

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 0.602 \text{ Hz} \tag{16}$$

(b) Critical damping occurs when $\omega = 0$ or

$$\frac{b_c}{2m} = \sqrt{\frac{k}{m}} \tag{17}$$

$$b_c = 2\sqrt{km} = 1.1 \text{ kNs/m} \tag{18}$$

Problem 4. *Show that the all functions of the form $y(x,t) = f(x \pm vt)$ for any function $f(z)$ satisfy the linear wave equation (Equation 13.19)*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} . \tag{19}$$

Taking the partial derivatives with respect to space

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial}{\partial x}(x \pm vt) = \frac{\partial f}{\partial z} \tag{20}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial z^2} \cdot \frac{\partial z}{\partial x} = \frac{\partial^2 f}{\partial z^2} \cdot \frac{\partial}{\partial x}(x \pm vt) = \frac{\partial^2 f}{\partial z^2} , \tag{21}$$

where we have used the chain rule

$$\frac{\partial}{\partial x}\left(f(z(x))\right) = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} \tag{22}$$

with $z(x) = x \pm vt$.
Taking the partial derivitives with respect to time

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} = \frac{\partial f}{\partial z} \cdot \frac{\partial}{\partial t}\left(x \pm vt\right) = \pm v \frac{\partial f}{\partial z} \tag{23}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\pm v \frac{\partial f}{\partial z}\right) = \pm v \frac{\partial^2 f}{\partial z^2} \cdot \frac{\partial z}{\partial t} = \pm v \frac{\partial^2 f}{\partial z^2} \cdot \frac{\partial}{\partial t}\left(x \pm vt\right) = (\pm v)^2 \frac{\partial^2 f}{\partial z^2} = v^2 \frac{\partial^2 f}{\partial z^2} \; . \tag{24}$$

So

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{25}$$

which is what we set out to show.

Problem 5. *Jack and Jill are broadcasting kazoo music from their treehouse to a picnic below using a tin can telephone (`http://en.wikipedia.org/wiki/Tin_can_telephone`). Their transmitting string weighs $m = 140$ g, is $L = 30$ m long, and is stretched to a tension of $T = 45$ N. At what amplitude must Jack and Jill vibrate their end of the string to drive the far can at $P = 1$ W while playing the musical note A at $f = 440$ Hz?*

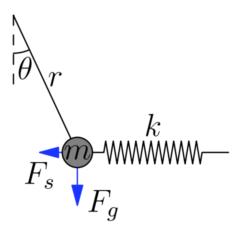
From Equation 13.23 we know that energy transfer in a sinusoidally oscillating string follows

$$P = \frac{1}{2}\mu\omega^2 A^2 v \tag{26}$$

The mass density of the string is $\mu = m/L \approx 4.667$ g/m, the speed of propagation is $v = \sqrt{T/\mu} \approx 98.20$ m/s, and the angular velocity of vibration is $\omega = 2\pi f \approx 2765$ rad/s. Solving the power formula for the amplitude we have

$$A = \frac{1}{\omega}\sqrt{\frac{2P}{\mu v}} \approx 0.76 \text{ mm} \tag{27}$$

Problem 6. BONUS PROBLEM. *Find the resonant frequency in Hz of the sprung pendulum for small θ on both Earth and the Moon. The mass of the bob is $m = 2.3$ kg, the length of the light rod is $r = 3.0$ m, and the spring constant is $k = 1.4$ N/m. The system is at equilibrium when the pendulum rod is vertical.*
Hints: drawing a free body diagram may help determine the restoring forces. You will need to use the small angle approximation.



The spring is stretched or compressed by $x \approx r\sin(\theta)$ where the approximation is exact in the limit of small angles. The total force is the sum of the spring force F_s and the gravitation force F_s acting on the bob. The portion of this total force that is tangent to the bob's path is

$$\sum F_{\text{tan}} = F_s\cos(\theta) - F_g\sin(\theta) = -kx\cos(\theta) - mg\sin(\theta) \approx -kr\sin(\theta)\cos(\theta) - mg\sin(\theta) = -\left[kr\cos(\theta) + mg\right]\sin(\theta) \tag{28}$$

$$\approx -\left[kr + mg\right] \cdot \theta \; , \tag{29}$$

where we have used the small angle approximation again in the last step. We also know from Newton's laws that

$$\sum F_{\text{tan}} = ma_{\text{tan}} = m\frac{\mathrm{d}^2 x_{\text{tan}}}{\mathrm{d}t^2} = mr\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \; . \tag{30}$$

Combining these two formulas for $\sum F$ we have

$$mr\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -kr + mg \cdot \theta \tag{31}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{kr + mg}{mr} \cdot \theta = -\left(\frac{k}{m} + \frac{g}{r}\right) \cdot \theta \; . \tag{32}$$

Looking at the last form, we see that it looks a lot like the equation of motion for simple harmonic motion

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta \ , \tag{33}$$

and we see that the equations are equal when

$$\frac{k}{m} + \frac{g}{r} = \omega^2 \ . \tag{34}$$

Plug in to the frequency formula for a simple harmonic oscillator, we have

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} + \frac{g}{r}} \ . \tag{35}$$

On Earth $g = 9.8$ m/s^2, and on the Moon $g = 1.6$ m/s^2, so

$$f_{\mathrm{Earth}} = \frac{1}{2\pi}\sqrt{\frac{1.4}{2.3} + \frac{9.8}{3.0}} = 0.31 \text{ Hz} \tag{36}$$

$$f_{\mathrm{Moon}} = \frac{1}{2\pi}\sqrt{\frac{1.4}{2.3} + \frac{1.6}{3.0}} = 0.17 \text{ Hz} \ . \tag{37}$$