Lorentz Velocity Transformations

$$u_{x'} = \frac{dx'}{dt'} \quad \begin{array}{l} \nearrow dx' = \gamma\,(dx - v\,dt) \\ \searrow dt' = \gamma\left(dt - \frac{v}{c^2}dx\right) \end{array}$$

$$u_{x'} = \frac{dx - v\,dt \;/:dt}{dt - v/c^2\,dx \;/:dt} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} =$$

$$= \frac{u_x - v}{1 - v/c^2\,u_x}$$

$$\frac{dx}{dt} = u_x$$

$$\boxed{u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u'_{y,z} = \frac{u_{y,z}}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}}$$

γ^{-1} only in $u'_{y,z}$

The speed of light in two different frames S & S':

in S: $u_x = c$

in S': $u'_x = \dfrac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \overset{u_x = c}{=}$

$$= \frac{c - v \,|: c \times c}{1 - \frac{v}{c}} = c \cdot \frac{\cancel{1 - \frac{v}{c}}}{\cancel{1 - \frac{v}{c}}}$$

$$= c$$

$\Rightarrow$ The speed of light in vacuum is $c = 3 \times 10^8$ m/s in ALL inertial frames.

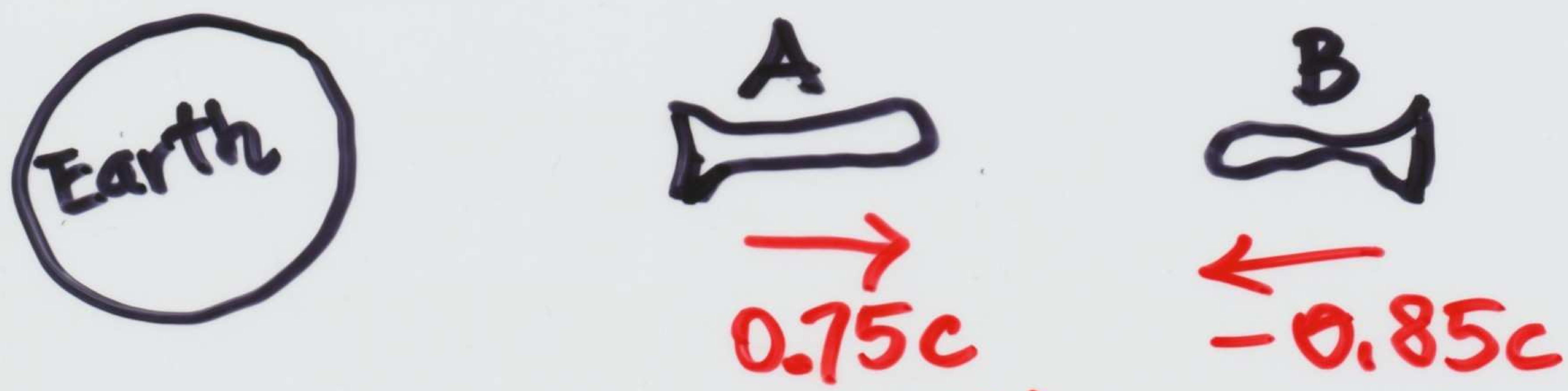
Observers in S & S' agree on:

- → relative velocity of S & S'
- → speed of light
- → simultaneity of 2 events that happen at fixed (x, y, z, t)

Observers in S & S' DO NOT AGREE on:

- time duration of an event that happen in S (or S') at the same spatial position
- Length of the object at rest in one of the frames (S or S') only
- the velocity components of a moving object
- simultaneity of two events that happen at different positions

Example:



(velocities as observed from the Earth)

How fast is the spaceship B approaching A as observed by A?

29-32

→ identify S and S'

S : Earth

S': spaceship A

→ calculate the velocity of B with respect to A (S')

$u_x = -0.85c$

$v = +0.75c$

Lorentz velocity transformation:

$$u_x' = \frac{u_x - v}{1 - u_x v/c^2} =$$

$$= \frac{-0.85c - 0.75c}{1 + 0.85 \cdot 0.75 \cancel{c^2}/\cancel{c^2}} =$$

$$= -\underline{\underline{0.98c}} < c$$

Relativistic Momentum and Relativistic Newton's Law

→ the mass of an object, m, is INVARIANT and equal to the value measured at rest

→ the total momentum of an isolated system of particles is CONSERVED

Problem with a classical definition of the momentum:

$$\vec{p} = m\vec{u}$$

→ consider a collision between two particles in S and S'

→ use $\vec{p}_A = m_A \vec{u}_A$ & $\vec{p}_B = m_B \vec{u}_B$

$\vec{p}_A + \vec{p}_B$ conserved in S

⇒ NOT conserved in S'

Relativistic momentum of a particle moving with a speed u (velocity $\vec{u}$):

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma m\vec{u}$$

→ when $u/c \to 0$ $(u \ll c)$

$$\Rightarrow \vec{p} \to m\vec{u}$$
(classical $\vec{p}$)

Relativistic force:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

what happens under the action of a constant force?

$$\rightarrow \vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \qquad \left(\vec{F} = \frac{d\vec{p}}{dt}\right)$$

$$\rightarrow \frac{d\vec{p}}{dt} = \frac{m}{(1-u^2/c^2)^{1/2}}\frac{d\vec{u}}{dt} - $$

$$- \frac{1}{\cancel{2}}\,\frac{-m\vec{u}\,\cancel{2}\vec{u}\;d\vec{u}/dt}{(1-u^2/c^2)^{3/2}\;c^2} =$$

$$= \frac{m\dot{\vec{u}}}{(1-u^2/c^2)^{1/2}}\left[1 + \frac{\cancel{m}\,u^2/\cancel{m}\,c^2}{1-u^2/c^2}\right]$$

$$= \underline{m\dot{\vec{u}}\,(1-u^2/c^2)^{-3/2}}$$

$$\rightarrow \vec{F} = \text{const.} \Rightarrow \dot{\vec{u}} \propto (1-u^2/c^2)^{+3/2}$$

$$\dot{\vec{u}} = \vec{a} \ldots \text{acceleration}$$

$u \rightarrow c \quad \dot{\vec{u}} = \vec{a} \rightarrow 0$

Relativistic Energy

kinetic energy versus rest energy

→ definition of K needs to be modified

work done by a net force on the particle = change in K

→ at $t=0$, $u=0$ & $K=0$

→ force $x_1 \rightarrow x_2$

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx =$$

$$= \int_{x_1}^{x_2} \frac{m}{\left(1-u^2/c^2\right)^{3/2}} \frac{du}{dt} \underbrace{\frac{dx}{dt}\cdot dt}_{=dx} =$$

$$= \int_0^u \frac{mu}{(1-u^2/c^2)^{3/2}}\,du =$$

$$= \int_0^{u/c} \frac{mc^2 \frac{u}{c}\, d\left(\frac{u}{c}\right)}{(1-u^2/c^2)^{3/2}} =$$

$u = \frac{dx}{dt}$

&

$\frac{du}{\cancel{dt}}\cancel{dt} = du$

$$= \frac{1}{2}mc^2 \int_0^{(u/c)^2} \frac{dz}{(1-z)^{3/2}} =$$

$z = \frac{u^2}{c^2}$

$dz = \frac{2u\,du}{c^2}$

$$= +\frac{1}{\cancel{2}}mc^2 \cdot \cancel{2}\Big[(1-z)^{-1/2}\Big]_0^{(u/c)^2} =$$

$$= +mc^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right] =$$

$$= mc^2 \cdot \gamma - mc^2$$

$$K = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2 = mc^2(\gamma - 1)$$

At small speeds $u \ll c$:

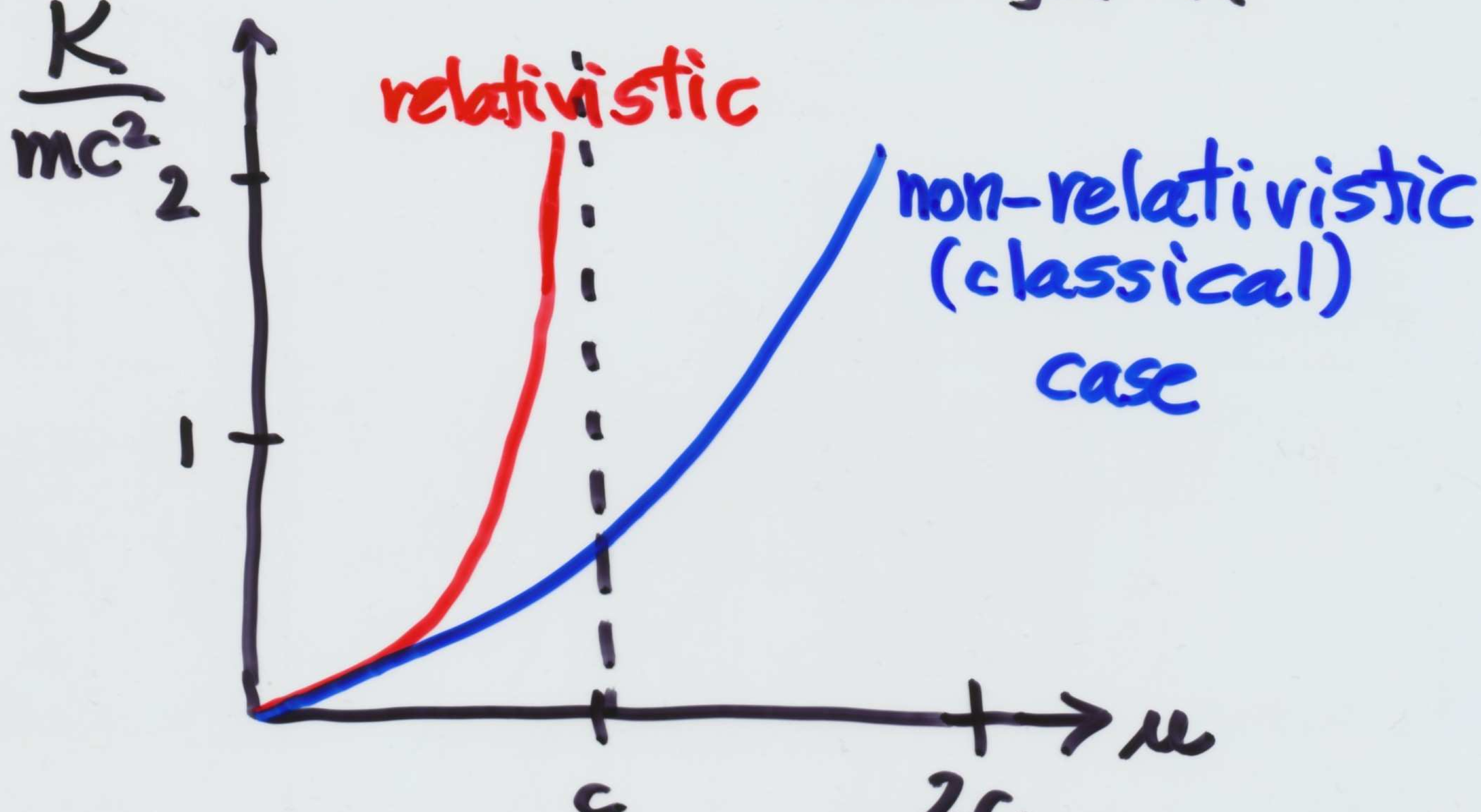
$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx$$

$$\approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \dots$$

Thus:

$$K = (\gamma - 1) mc^2 \approx \frac{1}{2} m\cancel{c^2} \frac{u^2}{\cancel{c^2}}$$

$K \approx \frac{1}{2} m u^2$ classical expression for K



→ rest energy E_R

$$E_R = mc^2$$

mass ↦ manifestation of energy

→ total energy E

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

→ $E = K + E_R$

Total energy is a sum of the rest and kinetic energies.

Energy – relativistic momentum relationship

$$E = \gamma mc^2 \qquad p = \gamma mu$$

$$E = \gamma mc^2 \qquad cp = \gamma m u c$$

$\downarrow$ square $\downarrow$

$$E^2 = \frac{(mc^2)^2}{1 - u^2/c^2} \qquad c^2p^2 = \frac{(mc^2)^2 \frac{u^2}{c^2}}{1 - u^2/c^2}$$

$$(-)$$

$$E^2 - c^2p^2 = \frac{(mc^2)^2}{1 - u^2/c^2} - \frac{(mc^2)^2 \, u^2/c^2}{1 - u^2/c^2}$$

$$= (mc^2)^2 \underbrace{\left[\frac{1}{1 - u^2/c^2} - \frac{u^2/c^2}{1 - u^2/c^2} \right]}_{=1}$$

$$E^2 = c^2p^2 + (mc^2)^2$$

photons $(m=0)$: $E = cp$

Quiz: Consider three particles:

(1) $E_R = E_0$ & $E_T = 2E_0$

(2) $E_R = 2E_0$ & $E_T = 4E_0$

(3) $E_R = E_0$ & $E_T = 3E_0$

Rank particles: $E_T = \underbrace{E_R}_{mc^2} + K$

→ according to mass:

$m_1 < m_2$ $\quad$ $m_2 > m_3$

$m_1 = m_3$

→ according to K: $K = E_T - E_R$

$K_1 < K_2$ $\quad$ $K_2 = K_3$

$K_1 < K_3$

→ according to speed: $E_T = \gamma E_R$

$u_1 \quad u_2$ $\quad$ $u_2 \quad u_3$

$u_1 \quad u_3$

Example: Consider a proton moving with a relativistic speed.

1 MeV $= 10^6$ eV

1 GeV $= 10^9$ eV

1 KeV $= 10^3$ eV

$m_p = 1.67 \times 10^{-27}$ kg

$c = 3 \times 10^8$ m/s

(A) $E_R = ?$

$$E_R = m_p c^2 = \frac{1.67 \times 10^{-27}\ 9 \times 10^{16}}{1.6 \times 10^{-19}} \text{eV}$$

$$= 9.38 \times 10^8 \text{eV} = \underline{\underline{938 \text{ MeV}}}$$

(B) $\underline{E_T = 3E_R} \Rightarrow u = ?$

$$E_T = \gamma \cdot E_R \Rightarrow \gamma = 3$$

$$\left(\frac{1}{\sqrt{1 - u^2/c^2}}\right)^2 = 9 \Rightarrow 1 - u^2/c^2 = \frac{1}{9}$$

$$u = \frac{\sqrt{8}}{3} c \approx 2.83 \times 10^8 \text{m/s}$$

(C) $K = E_T - E_R = 2E_R = \underline{\underline{1.88 \text{ GeV}}}$

Mass-Energy equivalence

→ total energy $E = \gamma mc^2$

→ particle at rest

$$\gamma \to 1 \quad E \to E_R = mc^2$$

→ E_R needs to be included in ENERGY CONSERVATION laws

Nuclear processes: fission(1) & fusion(2)

(1) $^{235}U \mapsto$ 2 lighter nuclei & neutrons

$$\Delta mc^2 = m_U c^2 - (m_1 + m_2 + \dots) c^2 > 0$$

$$\Delta mc^2 \mapsto K \text{ of products}$$

(2) 2 deuterium atoms ↦ 1 He-atom

$\Delta m = 4.29 \times 10^{-29}$ kg

$\Delta mc^2 = 23.9$ MeV

1 g deuterium → He ↦ 10^{12} J

29-44

The Compton Effect

→ Einstein proposed that a photon carries a momentum

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

(remember: photon is massless so $p = \gamma m u$ → ? not working)

$\gamma \to \infty$ as $v \to c$ & $m \to 0$

→ Compton: if a photon has a non-zero momentum, it can collide with an electron

Compton & co-workers studied scattering of X-rays from electrons, using one input X-ray frequency f_0 ⇒ observed ONLY ONE f output frequency

Classical wave theory:

→ X-ray beam → radiation pressure → force → accelerated e^-

→ oscillating $\vec{E}$ in the EM wave (X-rays) → oscillating e^-

→ f_0 radiation absorbed by MOVING e^- & e^- re-radiates X-ray as it moves ⇒ Doppler shifted emitted radiation

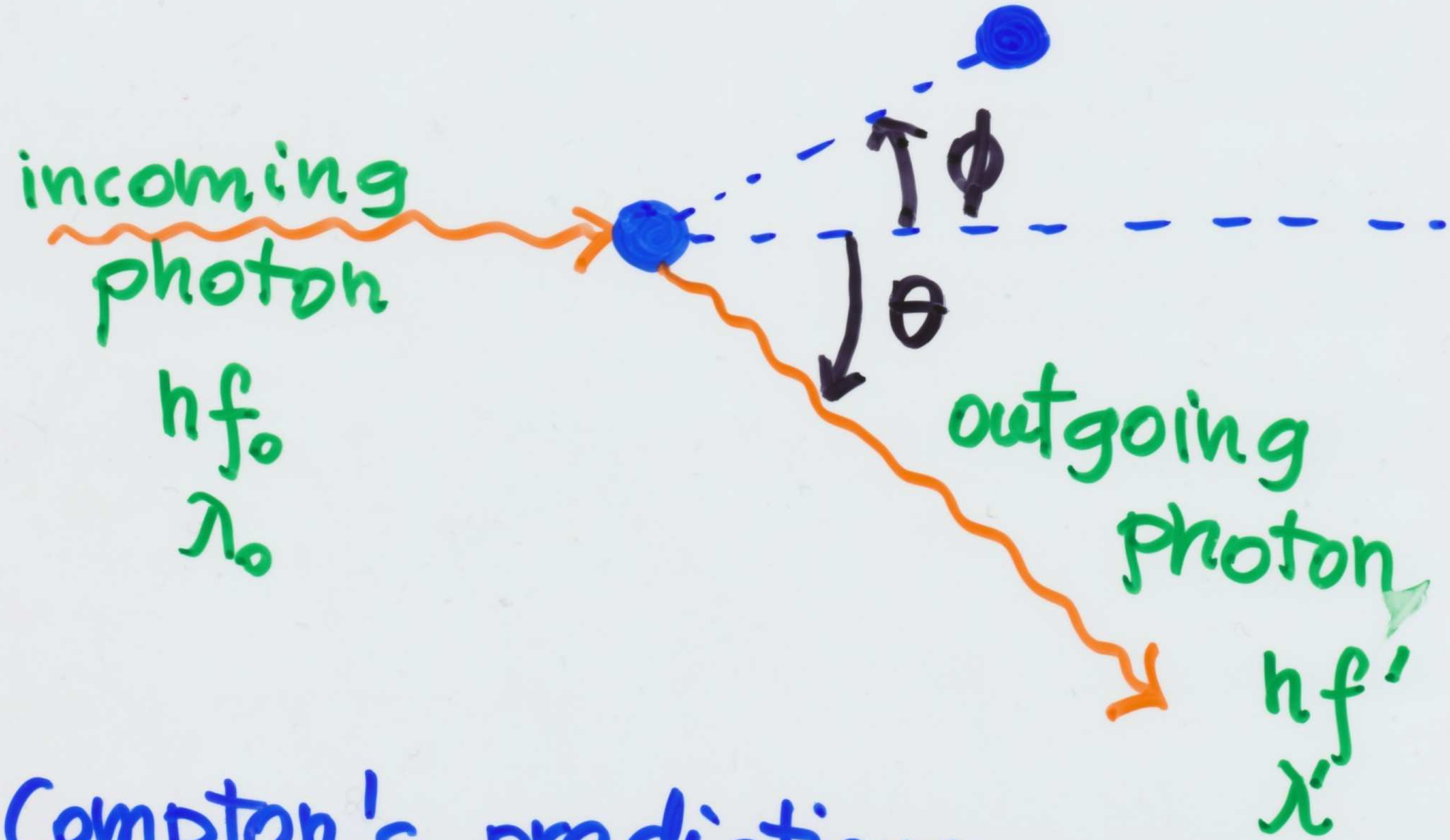
→ different e^- move at different velocities (speeds)

⇓

re-emitted radiation should be composed of a range of frequencies f ⇒ NOT true

Quantum physics explanation:

- → photons treated as particles
- → photon–electron collision
  - → momentum & energy conserved



Compton's predictions:

→ λ' depends on θ

$$\lambda' = \lambda_0 + \frac{h}{m_e c}(1 - \cos\theta)$$

→ Compton's wavelength λ_c

$$\lambda_c = \frac{h}{m_e c} = 0.00243\,\text{nm}$$

for the electron

→ Compton used as the initial wavelength $\lambda_0 = 0.071\,\text{nm}$

↓

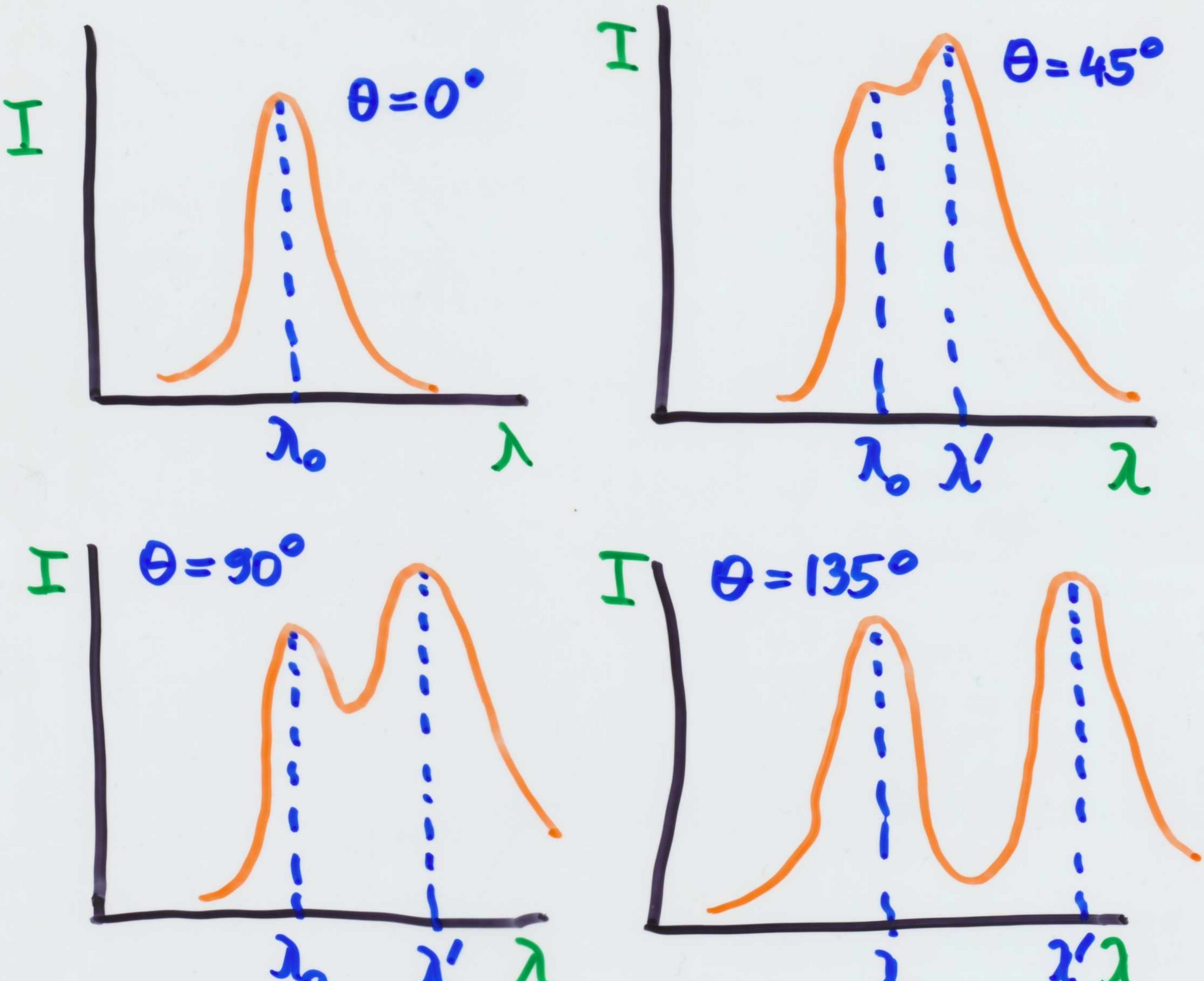
at $\theta = 180°$ (back scattering) the largest change in the wavelength:

$$\lambda' - \lambda_0 = \lambda_c \cdot 2 = 0.00486\,\text{nm}$$

$$\frac{\Delta\lambda}{\lambda_0} = \frac{0.00486\,\cancel{\text{nm}}}{0.071\,\cancel{\text{nm}}} \doteq \underline{\underline{0.07}}$$

→ very small change of λ

→ in hospitals & radiology labs: x-rays are Compton scattered from the human body in all directions ⇒ damage



→ effect stronger for very short wavelengths λ_0

Photons versus EM waves
(particle-wave duality)

→ photons: energy $E = hf$

momentum $p = \frac{E}{c}$

PARTICLES?

→ EM waves: interference

diffraction

WAVES?

⇓

Light (& x-rays, UV, IR and other parts of EM waves) has dual nature

A complete understanding requires integrating particle & wave picture into one.

The Wave Properties of Particles

Do particles have wave properties?

De Broglie's postulate:

a particle (electron, for exam.) in motion exhibits both particle & wave properties

1929, Nobel Prize

→ photon:

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

→ electron or other particle with non-zero mass:

$$\lambda = \frac{h}{p} = \frac{h}{mv} \quad \& \quad f = \frac{E}{h}$$

non-relativistic (classical) momentum $p = mv$