# PHYSICS 201 W03 EXAM 3 SOLUTIONS

### PROBLEM 1 (SPECIAL THEORY OF RELATIVITY)

An advanced spacecraft travels past Earth and Mars in a straight line at speed $v = 0.8c$ at a time when Earth and Mars are $2.4 \times 10^{11} m$ apart. The distance is measured in the fixed frame of reference, in which the Sun, Earth, and Mars are at rest. (We neglect any motion of the planets with respect to each other and the Sun during this short time.)

**(A) In the fixed reference frame, how long does it take for the spacecraft to go from Earth to Mars?**

The measurement of the distance between Earth and Mars is taken by a "ruler" at rest in the Earth and Mars frame, and so is the proper length: $L_p = 2.4 \times 10^{11} m$. Thus if the velocity of the ship is $v = 0.8c$ in the Earth-Mars frame, the ship will take

$$\Delta t = \frac{L_p}{v} = \frac{2.4 \times 10^{11} s}{0.8(3 \times 10^8 m/s)} = 1000s$$

**(B) In the frame in which the spacecraft is at rest, how fast are the Earth and Mars moving?**

The ship will see the Earth and Mars moving at $v = 0.8c$.

**(C) In the spacecraft frame, how far apart are the Earth and Mars?**

Since we know that the measurement given, $2.4 \times 10^{11} m$ is the proper length (the "ruler" by which this was measured is at rest in the Earth-Mars frame), then we know that the ship will see the distance between Earth and Mars contracted:

$$L = \frac{L_p}{\gamma} = \frac{2.4 \times 10^{11} m}{5/3} = \frac{3}{5}(2.4 \times 10^{11} m) = 1.44 \times 10^{11} m$$

Where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.8c)^2}{c^2}}} = \frac{5}{3}$$

**(D) In the spacecraft frame, how much time elapses between the Earth passing by, and Mars passing by?**

We can solve this two ways. We know that the proper time measurement is taken on the space ship because the two events by which we measure the time interval, the time the space ship leaves Earth and the time the ship reaches mars, appear to happen at the same place in the ship frame–that is, it looks to an observer on the ship like Earth is going away from them and mars is going towards them. Thus they measure proper time and we can use the time dilation equation:

$$\Delta t_p = \frac{\Delta t}{\gamma} = \frac{3}{5}1000s = 600s$$

We can also use the same method as in part A:

$$\Delta t' = \frac{L}{v} = \frac{1.44 \times 10^{11}}{0.8(3 \times 10^8 m/s)} = 600s$$

PROBLEM 2 (RELATIVISTIC MOMENTUM AND ENERGY)

Electrons and protons are accelerated through a voltage difference of $10^9 V$. The rest energies of the electron and proton are $0.511 MeV$ and $938 MeV$, respectively.

### (A) Calculate the relativistic momenta of electrons and protons

The rest energies are $m_e c^2 = 0.511 MeV$ for the electron, and $m_p c^2 = 938 MeV$ for the proton.

When a charged particle is accelerated through a voltage difference of $10^9 V$, its kinetic energy will be $Q(10^9 V)$ where Q is the charge of the particle. In this case, both the proton and the electron have a charge $e$ (the proton is $+e$ and the electron is $-e$) so both particles will end up with kinetic energy $10^9 eV$ (remember in the photoelectric effect how we related the kinetic energy with the stopping potential– this is the same idea). We can thus use the equation $E^2 = c^2 p^2 + (mc^2)^2$:

$$E_e = K_e + m_e c^2 \qquad E_e^2 = c^2 p_e^2 + (m_e c^2)^2$$
$$E_p = K_p + m_p c^2 \qquad E_p^2 = c^2 p_p^2 + (m_p c^2)^2$$

Thus

$$cp_e = \sqrt{E_e^2 - (m_e c^2)^2} = \sqrt{K_e^2 + 2m_e c^2 K_e} = K_e \sqrt{1 + \frac{2m_e c^2}{K_e}}$$
$$= 10^9 eV \sqrt{1 + \frac{2(0.511 \times 10^6 eV)}{10^9 eV}} = 1.0005 \times 10^9 eV$$

For the proton we get the same result:

$$cp_p = K_p \sqrt{1 + \frac{2m_p c^2}{K_p}} = 10^9 eV \sqrt{1 + \frac{2(938 \times 10^6 eV)}{10^9}} = 1.696 \times 10^9 eV$$

**Alternative solution:**

$$K_e = 10^9 eV = (\gamma-1)m_e c^2 = (\gamma-1)0.511\times10^6 eV \implies \gamma = 1957.95 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \implies v = 0.99999987c$$

Thus $cp_e = c(\gamma m v) = 1957.95(0.99999987)m_e c^2 = 1.0005 \times 10^9 eV$

Similarly for the proton,

$$K_p = 10^9 eV = (\gamma-1)m_p c^2 = (\gamma-1)938\times10^6 eV \implies \gamma = 2.066 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \implies v = 0.875c$$

Thus $cp_p = c(\gamma m v) = 2.066(0.875)m_p c^2 = 1.696 \times 10^9 eV$

### (B) Calculate the total energy of these electrons and protons.

$$E_e = K_e + m_e c^2 = 1000.511 \times 10^6 eV$$
$$E_p = K_p + m_p c^2 = 1938 \times 10^6 eV$$

PROBLEM 3 (COMPTON EFFECT)

In a Compton scattering event, the *scattered* photon has an energy
of 150 keV and the recoiling *electron* has a kinetic energy of 40 keV.
Assume that before the scattering the electron is at rest.

**(A) Find the wavelength (in nanometers) and the energy (in keV) of
the *incident* photon.**

*Energy conservation* tells us that the total energy of this system before the
electron is hit by the photon is the same as the total energy of the system after the
photon hits the electron.

Ignoring the rest energy, because it doesn't change, we have

$$
\begin{aligned}
E_i &= E_f \\
\frac{hc}{\lambda_i} + 0 &= \frac{hc}{\lambda_f} + K_e \\
\frac{hc}{\lambda_i} &= 150 \times 10^3 eV + 40 \times 10^3 eV \\
\frac{1240 eV(nm)}{\lambda_i} &= 190 \times 10^3 eV \\
\frac{1240 eV(nm)}{190 \times 10^3 eV} &= \lambda_i \\
0.0065 nm &= \lambda_i
\end{aligned}
$$

**(B) First find the wavelength (in nanometers) of the *scattered* photon
and then calculate the scattering angle.**

$$
\frac{hc}{\lambda_f} = 150 \times 10^3 eV \implies \lambda_f = \frac{hc}{150 \times 10^3 eV} = \frac{1240 eV(nm)}{150 \times 10^3 eV} = 0.0083 nm
$$

Now we can apply the Compton shift equation:

$$
\Delta \lambda = \lambda_f - \lambda_i = 0.0083 nm - 0.0065 nm = 0.0018 = \lambda_c(1 - \cos \theta)
$$

We solve for $\cos \theta$:

$$
\cos \theta = 1 - \frac{\lambda_f - \lambda_i}{\lambda_c} = 1 - \frac{0.0018 nm}{0.0024 nm} = 0.25
$$

Take the inverse-cosine to find $\theta = 72.5°$.

### PROBLEM 4 (DEBROGLIE WAVELENGTH)

The electrons in Davisson-Germer experiment had a kinetic energy of 54 eV.

**(A) Calculate the DeBroglie wavelength of these electrons.**

The kinetic energy of the electrons is much less than the rest energy of the electron, $54eV \ll 0.511^6 eV$, so we can use the classical formula which relates kinetic energy ($K = mv^2/2$i) to momentum ($p = mv$):

$$K = \frac{1}{2}mv^2 = \frac{1}{2m}m^2v^2 = \frac{p^2}{2m} \implies p = \sqrt{2mK}$$

Using the DeBroglie wave equation, we have

$$
\begin{aligned}
\lambda_e &= \frac{h}{p} \\
&= \frac{hc}{c\sqrt{2m_e K}} \\
&= \frac{hc}{\sqrt{2m_e c^2 K}} \\
&= \frac{1240eV(nm)}{\sqrt{2(0.511 \times 10^6 eV)54eV}} \\
&= \frac{1240nm}{7428.86} \\
&= 0.167nm
\end{aligned}
$$

**(B) What would be the DeBroglie wavelength of protons of the same kinetic energy of $54eV$?**

Again, since the kinetic energy is much less than the rest energy of protons, we can use the classical formula for kinetic energy and, as in part A:

$$\lambda_p = \frac{hc}{\sqrt{2m_p c^2 K}} = \frac{1240eV(nm)}{\sqrt{2(938 \times 10^6)54eV}} = 0.003896nm$$

And so we note that the more massive objects have much smaller wavelengths than less massive objects with the same kinetic energy.