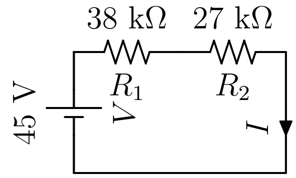
Recitation 8

Chapter 19

Problem 58. A 45 V battery of negligable internal resistance is connected to a 38 kΩ and a 27 kΩ resistor in series. What reading will a voltmeter, of internal resistance 95 kΩ, give when used to measure the voltage across each resistor? What is the percent inaccuracy due to meter resistance for each case?

Case 1:
The original situation looks like



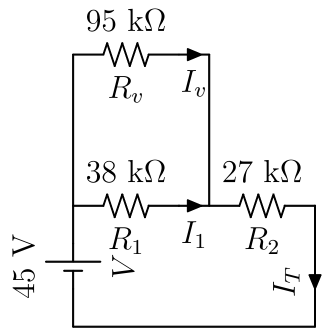
Using Kirchoff's loop rule

$$V - IR_1 - IR_2 = 0$$
$$I = \frac{V}{R_1 + R_2}$$

so

$$V_1 = IR_1 = \frac{VR_1}{R_1 + R_2} \approx 26.3 \text{ V}$$
$$V_2 = IR_2 = \frac{VR_2}{R_1 + R_2} \approx 18.7 \text{ V}$$

Case 2:
With the voltmeter across R_1 we have



Using our formula for resistors in parallel, we can bundle R_v and R_1 into a single resistor R_1', where

$$R_1' = \left(\frac{1}{R_1} + \frac{1}{R_v}\right)^{-1} = 27.14285714\ldots \text{ k}\Omega$$

Once we've done that, we have the same situation as in Case 1, but with

$$R_1 \to R_1'$$
$$I \to I_T$$

so

$$V_1' = \frac{VR_1'}{R_1' + R_2} = \frac{V\left(\frac{1}{R_1} + \frac{1}{R_v}\right)^{-1}}{\left(\frac{1}{R_1} + \frac{1}{R_v}\right)^{-1} + R_2} \approx 22.6 \text{ V}$$

$$\frac{V_1'}{V_1} = \frac{R_1'(R_1 + R_2)}{R_1(R_1' + R_2)} = \frac{\left(\frac{1}{R_1} + \frac{1}{R_v}\right)^{-1} \cdot (R_1 + R_2)}{R_1 \cdot \left[\left(\frac{1}{R_1} + \frac{1}{R_v}\right)^{-1} + R_2\right]} = \frac{\left(\frac{R_v + R_1}{R_1 R_v}\right)^{-1} \cdot (R_1 + R_2)}{R_1 \cdot \left[\left(\frac{R_v + R_1}{R_1 R_v}\right)^{-1} + R_2\right]} = \frac{\frac{R_1 R_v}{R_v + R_1} \cdot (R_1 + R_2)}{R_1 \cdot \left(\frac{R_1 R_v}{R_v + R_1} + \frac{R_2(R_v + R_1)}{R_v + R_1}\right)}$$

$$= \frac{R_1 \frac{R_v(R_1 + R_2)}{R_v + R_1}}{R_1 \cdot \left(\frac{R_1 R_v + R_2(R_v + R_1)}{R_v + R_1}\right)} = \frac{R_v R_1 + R_v R_2}{R_1 R_v + R_2 R_v + R_1 R_2} = \frac{R_v(R_1 + R_2)}{R_v(R_1 + R_2) + R_1 R_2}$$

$$= 0.8575\ldots$$

Obviously, we could plug in known numbers and solve for $\frac{V_1'}{V_1}$ after the first equality above, but crunching through some simplifying algebra reveals the pretty spectacular final form, from which you can trivially see that the fractional error in Case 3 will be the same as that for Case 2.
Finally the percent error is given by

$$\text{Error}_1 = 1 - \frac{V_1'}{V_1} = 0.1425\ldots \approx 14\%$$

Case 3:
With the voltmeter across R_2 we have the same situation as Case 1, but with

$$X_1 \leftrightarrow X_2$$

for any symbol X (i.e. $R_1 \leftrightarrow R_2, \ldots$). However, the equation for error in V_1' is not effected by this exchange, so

$$\text{Error}_2 = \text{Error}_1 \approx 14\%$$

The voltage V_2' measured is given by

$$V_2' = \frac{VR_2'}{R_2' + R_1} = \frac{V\left(\frac{1}{R_2} + \frac{1}{R_v}\right)^{-1}}{\left(\frac{1}{R_2} + \frac{1}{R_v}\right)^{-1} + R_1} \approx 16.0 \text{ V}$$