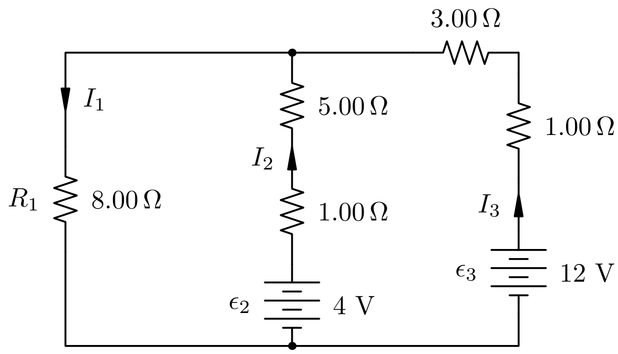
Recitation 6

Chapter 21

Problem 35. Determine the current in each branch of the circuit shown in Figure P21.35.



Let I_1 be the current on the left branch (going down), I_2 be the current on the middle branch (going up), and I_3 be the current on the right branch (going up). From Kirchhoff's junction rule, we know.

$$I_1 = I_2 + I_3 \tag{1}$$

Let $\epsilon_2 = 4.00$ V be the voltage across the middle battery, and $\epsilon_3 = 12.0$ V be the voltage across the right battery.
Using our knowledge of series resistors, we find

$$R_1 = 8.00\Omega \tag{2}$$
$$R_2 = 5.00\Omega + 1.00\Omega = 6.00\Omega \tag{3}$$
$$R_3 = 3.00\Omega + 1.00\Omega = 4.00\Omega \tag{4}$$

We can use Ohm's law to find the voltage drops across them in the direction of their current.
Now using Kirchhoff's loop rule on the left-center and left-right loops respectively we have

$$0 = \epsilon_2 - I_2R_2 - I_1R_1 \tag{5}$$
$$0 = \epsilon_3 - I_3R_3 - I_1R_1 \tag{6}$$

So we have our three equations relating our unknown currents. If you're comfortable with linear algebra (take a look at my linear algebra intro if you want to get comefortable), you can express these as a matrix

$$\begin{pmatrix} 0 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ R_1 & R_2 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{7}$$

Inverting the 3x3 matrix, we get

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 8.00\Omega & 6.00\Omega & 0 \\ 8.00\Omega & 0 & 4.00\Omega \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -0.2308 & 0.0385 & 0.0577 \\ 0.3077 & 0.1154 & -0.0769 \\ 0.4615 & -0.0769 & 0.1346 \end{pmatrix} \begin{pmatrix} 0 \\ 4.00 \\ 12.0 \end{pmatrix} = \begin{pmatrix} 0.8462 \\ -0.4615 \\ 1.3077 \end{pmatrix} \text{ A} \tag{8}$$

Where $I_2 < 0$ indicates that current actually flows in the opposite direction to what we expected.
If you're not comfortable with linear algebra, you can solve the equations using your method of choice. If no methods make sense to you, come talk to me or get someone else to teach you one. If you want to double check your algebra, I work the solution out symbolically in my linear algebra introduction in traditional equation format as well as in matrix format.
The benifit of the linear algebra is that most graphing calculators can do the matrix inversion for you. On the TI-89, you can do

$$[-1, 1, 1; 8, 6, 0; 8, 0, 4] \rightarrow A \tag{9}$$
$$[0; 4; 12] \rightarrow I \tag{10}$$
$$A^{-1} * I \tag{11}$$
$$\begin{pmatrix} 0.8462 \\ -0.4615 \\ 1.3077 \end{pmatrix} \tag{12}$$

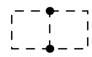
(I don't have a TI-89, so if this is wrong, let me know..., see my linear algebra introduction for TI-83+ rules).

Problem 38. The following equations describe an electric circuit:

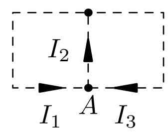
$$-(220\Omega)I_1 + 5.80\text{ V} - (370\Omega)I_2 = 0 \tag{13}$$
$$(370\Omega)I_2 + (150\Omega)I_3 - 3.10\text{ V} = 0 \tag{14}$$
$$I_1 + I_3 - I_2 = 0 \tag{15}$$

(a) Draw a diagram of the circuit. (b) Calculate the unknowns and identify the physical meaning of each unknown.

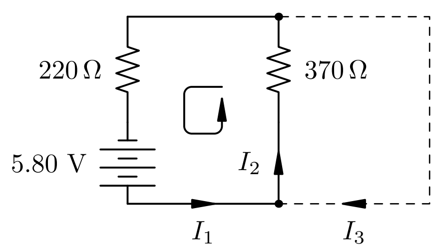
(a) Looking at the three equations, we see that the only unknowns are I_1, I_2, and I_3. That looks like a circuit with current in three branches.



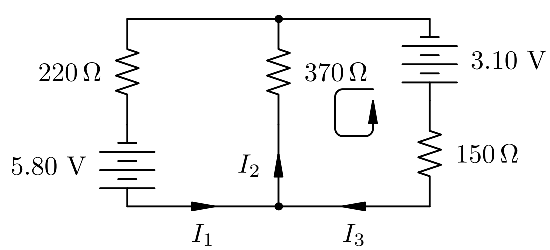
By looking at Eqn. 15 and identifying it with Kirchhoff's junction rule on junction A, we can get current directions.



Eqn. 13 looks like a Kirchhoff's loop rule involving only branches 1 and 2. The first term $-(220\Omega)I_1$ looks like a $V = IR$ resistor drop in the direction of the current on branch 1, so let's add a 220Ω resistor to branch 1. Because the voltage drops in our loop equation, we must be moving in the direction of the current. Continuing through the Eqn. 13, we see a constant voltage increase, which looks like we crossed a battery from the negative to positive side, so we'll add that onto branch 1 too. Finally, there is a $-(370\Omega)I_2$ drop which looks like crossing a resistor in the direction of the current on branch 2, so let's add a 370Ω resistor to branch 2.



Eqn. 14 looks like another Kirchhoff's loop rule, this time involving only branches 2 and 3. The first term $-(370\Omega)I_2$ looks like a resistor gain *against* the direction of the current on branch 2. We already have a 370Ω resistor to branch 2, so this term just tells us we're moving upstream against I_2. Continuing through the Eqn. 14, we see another voltage *gain* $(150\Omega)I_3$. If we're moving upstream on I_2, we'll also be moving upstream on I_3, so this voltage gain must be a 150Ω resistor on branch 3. The last term is a constant votage *drop*, which looks like we crossed a battery from the positive to negative side, so we'll add that onto branch 3 too.



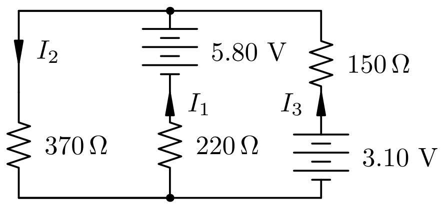
(b) Solve using your method of choice. With linear algebra:

$$\begin{pmatrix} 5.80\text{ V} \\ 3.10\text{ V} \\ 0 \end{pmatrix} = \begin{pmatrix} 220\Omega & 370\Omega & 0 \\ 0 & 370\Omega & 150\Omega \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{16}$$

Inverting the 3x3 matrix, we get

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0.0031 & -0.0022 & 0.3267 \\ 0.0009 & 0.0013 & -0.1942 \\ -0.0022 & 0.0035 & 0.4791 \end{pmatrix}^{-1} \begin{pmatrix} 5.80\text{ V} \\ 3.10\text{ V} \\ 0 \end{pmatrix} = \begin{pmatrix} 11.0 \\ 9.13 \\ -1.87 \end{pmatrix} \text{ mA} \tag{17}$$

With regular algebra, we can save ourselves a bit of work by noticing that this problem is the same as the one we just did (35)! Well, now we have a battery on the first branch and none on the second, and the batteries are facing down... If we flip the picture over and swap the first and second branches...



Alright, now it looks like the figure in Problem 35, except that the things labeled X_1 and X_2 are reversed. We can take our analytic solution to 35 (see the linear algebra notes) and exchange $1 \leftrightarrow 2$ giving

$$\frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_1}{R_1}}{\frac{R_2}{R_3} + \frac{R_2}{R_1} + 1} = I_2 = 9.13 \text{ mA} \tag{18}$$

$$\frac{\epsilon_1}{R_1} - \frac{1}{R_1} \frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_1}{R_1}}{\frac{1}{R_3} + \frac{1}{R_1} + \frac{1}{R_2}} = I_1 = 11.0 \text{ mA} \tag{19}$$

$$\frac{\epsilon_3}{R_3} - \frac{1}{R_3} \frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_1}{R_1}}{\frac{1}{R_3} + \frac{1}{R_1} + \frac{1}{R_2}} = I_3 = -1.87 \text{ mA} \tag{20}$$

Problem 42. A $C = 2.00\ \mu$F capacitor with an intial charge of $Q = 5.10\ \mu$C is discharged through an $R = 1.30\Omega$ resistor. (a) Calculate the current in the resistor $t_a = 9.00\ \mu$s after the resistor is connected across the terminals of the capacitor. (b) What charge remains on the capacitor after $t_b = 8.00\ \mu$s? (c) What is the maximum current in the resistor?

(a) The current through the entire circuit follows

$$I = \frac{Q}{RC} e^{-t/RC} \tag{21}$$

So

$$I(t_a) = \frac{5.10 \cdot 10^{-6} \text{ C}}{1.30\Omega \cdot 2.00 \cdot 10^{-6} \text{ F}} e^{\frac{-9.00 \cdot 10^{-6} \text{ s}}{1.30\Omega \cdot 2.00 \cdot 10^{-6} \text{ F}}} = 61.6 \text{ mA} \tag{22}$$

(b) The charge on the capacitor follows

$$q = Qe^{-t/RC} \tag{23}$$

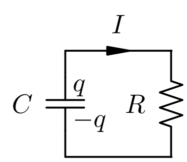
So

$$I(t_a) = 5.10 \cdot 10^{-6} \text{ C} e^{\frac{-8.00 \cdot 10^{-6} \text{ s}}{1.30\Omega \cdot 2.00 \cdot 10^{-6} \text{ F}}} = 235 \text{ nC} \tag{24}$$

(c) Plugging $t = 0$ into our equation from (a) we have

$$I_{max} = \frac{Q}{RC} = 1.96 \text{ A} \tag{25}$$

For those who are interested the derivation of Eqns. 21 and 23 is pretty straightforward. Consider the circuit



The current $I = -dq/dt$.
(Note: In the book it gives $I \equiv dQ/dt$ [Eqn. 21.2], but that dQ is the charge passing through a given cross section. Our q is the charge on the top capacitor plate. As charge leaves the top capacitor plate ($dq/dt < 0$), it passes through a point in the wire in the direction we've specified for I ($I > 0$), so $I = -dq/dt$. This is the point that tripped me up in Wednesday's recitation, right after I had warned about equating symbols without thinking about what they ment :p)
Using Kirchhoff's loop rule and the definition of capacitance $Q = CV$ and resistance $V = IR$, we have

$$\Delta V = +\frac{q}{C} - IR = 0 \tag{26}$$

$$\frac{q}{C} = IR = -\frac{dq}{dt} R \tag{27}$$

$$\frac{-dt}{RC} = \frac{dq}{q} \ . \tag{28}$$

Integrating both sides we have

$$\int \frac{-dt}{RC} = \frac{-1}{RC}\int dt = \frac{-1}{RC}(t+A) = \int \frac{dq}{q} = \ln(q) \ , \tag{29}$$

where A is some constant of integration because we were taking indefinite integrals. We want a function for $q(t)$, so we take e to the power of both sides

$$e^{-(t+A)/RC} = e^{-t/RC-A/RC} = A'e^{-t/RC} = e^{\ln(q)} = q \ , \tag{30}$$

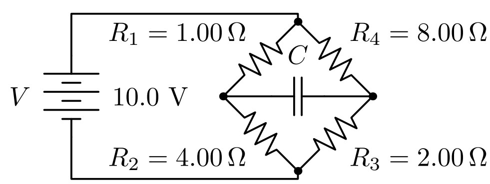
where $A' \equiv e^{-A/RC}$ is just another way of thinking about our arbitrary integration constant A. Looking at our initial condition, $q(t=0) = Q$, the initial charge on the capacitor, and comparing with our equation we have

$$q(t=0) = A'e^{-0/RC} = A' = Q \ , \tag{31}$$

so we can replace A' with Q to get Eqn. 23.
Eqn. 21 follows from Eqn. 23 using our $I = -dq/dt$:

$$I = -\frac{d}{dt}\left(Qe^{-t/RC}\right) = -Q\frac{d}{dt}e^{-t/RC}) = -Q\frac{-1}{RC}e^{-t/RC} = \frac{Q}{RC}e^{-t/RC} \tag{32}$$

Problem 45. The circuit in Figure P21.45 has been connected for a long time. (a) What is the voltage V_c across the capacitor? (b) If the battery is disconnected, how long does it take the capacitor to discharge to $V_c' = 1/10 \cdot V$?



Labeling the resistors counterclockwise from the upper left we have $R_1 = 1.00\Omega$, $R_2 = 4.00\Omega$, $R_3 = 2.00\Omega$, and $R_4 = 8.00\Omega$. Let $V = 10.0$ V be the voltage on the battery and $C = 1.00$ μF be the capacitance of the capacitor.
(a) Because the system has been running for a long time, the system must be close to equilibrium. Therefore, the current through the capacitor must be zero (otherwise the voltage across the capacitor would be changing, and you wouldn't be at equilibrium). The resistor bridge then reduces to two parallel circuits, and we can apply Ohm's law to determine V_c
Starting with the left side of the bridge (calling the current I_L),

$$V = I_L(R_1 + R_2) \ , \qquad I_L = \frac{V}{R_1 + R_2} \tag{33}$$

And on the right calling the current I_R

$$I_R = \frac{V}{R_3 + R_4} \tag{34}$$

So using Ohm's law to compute the voltage across the capacitor, we call the voltage on the bottom wire 0 and have the voltage V_L on the left at

$$V_L = I_L R_2 = \frac{VR_2}{R_1 + R_2} = 8 \text{ V} \tag{35}$$

And the voltage V_R to the right of the capacitor is

$$V_R = I_R R_3 = \frac{VR_3}{R_3 + R_4} = 2 \text{ V} \tag{36}$$

So the voltage across the capacitor is

$$V_c = V_L - V_R = 6 \text{ V} \tag{37}$$

(b) Once we remove the battery, we see that the capacitor discharges through two paths in parallel, $R_1 \to R_4$ and $R_2 \to R_3$. The eqivalent resistances of these two parallel branches (top and bottom) are

$$R_T = R_1 + R_4 \ , \qquad R_B = R_2 + R_3 \tag{38}$$

So the total equivalent resistance is

$$R = \left(\frac{1}{R_T} + \frac{1}{R_B}\right)^{-1} = 3.60\Omega \tag{39}$$

The voltage of a discharging capacitor depends on time according to

$$V_c' = V_c e^{-t/RC} \tag{40}$$

So using $V_c' = V_c/10$ we have

$$10 = \frac{V_c}{V_c'} = \frac{V_c}{V_c e^{-t/RC}} = e^{t/RC} \tag{41}$$

$$\ln(10) = \frac{t}{RC} \tag{42}$$

$$t = RC\ln(10) = 8.29\ \mu\text{s} \tag{43}$$

Problem 53. An electric heater is rated at $P_H = 1500$ W, a toaster at $P_T = 750$ W, and an electric grill at $P_G = 1000$ W. The three appliances are connected to a common $V = 120$ V household circuit. (a) How much current does each draw? (b) Is a circuit with a $V_{max} = 25.0$ A circuit breaker sufficient in this situation? Explain your answer.

(a) Using $P = IV$ we have

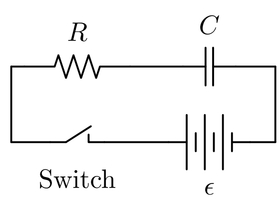
$$I_H = \frac{P_H}{V} = 12.5\ \text{A}\ , \qquad I_T = \frac{P_T}{V} = 6.25\ \text{A}\ , \qquad I_G = \frac{P_G}{V} = 8.33\ \text{A} \tag{44}$$

(b) If all the appliances are running together, the circuit draws

$$I = I_H + I_T + I_G = 27.1\ \text{A} \tag{45}$$

So you will be fine with a 25 A breaker unless you plan to run all three at the same time.

Problem 58. A battery with emf ϵ is used to charge a capacitor C through a resistor R as shown in Figure 21.25. Show that half the energy supplied by the battery appears as internal energy in the resistor and that half is stored in the capacitor.



The total current through the system is given by

$$I = I_o e^{-t/RC} = \frac{\epsilon}{R} e^{-t/RC} \tag{46}$$

Which allows us to compute the energy put out by the battery. Power is the time derivative of energy so

$$E_b = \int_0^\infty P \cdot dt = \int_0^\infty I\epsilon \cdot dt = \frac{\epsilon^2}{R}\int_0^\infty e^{-t/RC} \cdot dt = \frac{\epsilon^2}{R}\ -RCe^{-t/RC}\Big|_0^\infty = \frac{\epsilon^2}{R}\left(0 - (-RCe^0)\right) = C\epsilon^2 \tag{47}$$

Similarly for the energy absorbed by the resistor

$$E_r = \int_0^\infty P \cdot dt = \int_0^\infty I^2 R\epsilon \cdot dt = \frac{\epsilon^2}{R}\int_0^\infty e^{-2t/RC} \cdot dt = \frac{\epsilon^2}{R}\ \frac{-RC}{2} e^{-2t/RC}\Big|_0^\infty = \frac{E_b}{2} = \frac{1}{2}C\epsilon^2 \tag{48}$$

And we already know the energy stored in a capacitor with a voltage ϵ is

$$E_c = \frac{1}{2}C\epsilon^2 = \frac{E_b}{2} \tag{49}$$

So the battery energy splits evenly between the capacitor and the resistor, and we're done.