Linear algebra introduction

Solving systems of linear equations automatically

1 Writing matrix equations

All battery-resistor-current problems produce systems of *linear equations*. Linear equations are just a equations where the unknowns (usually the currents I_i) are never multiplied together. For example from Recitation 6, Problem 35, we have

$$0 = I_1 - I_2 - I_3 \tag{1}$$
$$0 = \epsilon_2 - I_2R_2 - I_1R_1 \tag{2}$$
$$0 = \epsilon_3 - I_3R_3 - I_1R_1 \ , \tag{3}$$

from Kirchhoff's laws. We're given everything except the currents, and no terms have two currents multiplied by each-other in them. We can rewrite these equations and adjust the spacing a bit to get

$$0 = -I_1 \ + \ I_2 \ + \ I_3 \tag{4}$$
$$\epsilon_2 = R_1I_1 + R_2I_2 + 0 \cdot I_3 \tag{5}$$
$$\epsilon_3 = R_1I_1 + 0 \cdot R_2 + R_3I_3 \ . \tag{6}$$

We can see that each line has the same format

$$b = a_1I_1 + a_2I_2 + a_3I_3 = \sum_{j=1}^{3} a_jI_j \ , \tag{7}$$

with constant a_is and bs.

We can rewrite the system of equations as a *matrix* equation

$$\begin{pmatrix} 0 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ R_1 & R_2 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{8}$$

Comparing this to Eqns. 4–6, we can see that all we've done is erased a few equals signs in the middle, drawn some big parenthesis, and dragged the unknown currents off by themselves, giving them their own parenthesis. We've also rotated order we wrote the currents in

$$\clubsuit I_1 + \diamondsuit I_2 + \heartsuit I_3 \longrightarrow \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \ . \tag{9}$$

Things look fairly different at first, so go back and compare Eqn. 8 with Eqns. 4–6 until you get comfortable with the changes.

2 Solving matrix equations

All the things we could do when we had regular equations, we can still do with the equations written in matrix form. We'll go through and solve these side by side so you can see a solution in action and become more familiar with the matrix notation. The only difference in solving strategy is that we keep all the symbols on the same side they started on, and add or subtract equations instead of plugging in.

$$\begin{array}{ccccccc} 0 & = & -I_1 & + & I_2 & + & I_3 \\ \epsilon_2 & = & R_1I_1 & + & R_2I_2 & + & 0 \\ \epsilon_3 & = & R_1I_1 & + & 0 & + & R_3I_3 \end{array} \qquad \begin{pmatrix} 0 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ R_1 & R_2 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{10}$$

Lets solve the middle equation for I_2 by dividing by R_2.

$$\begin{array}{ccccccc} 0 & = & -I_1 & + & I_2 & + & I_3 \\ \frac{\epsilon_2}{R_2} & = & \frac{R_1}{R_2}I_1 & + & I_2 & + & 0 \\ \epsilon_3 & = & R_1I_1 & + & 0 & + & R_3I_3 \end{array} \qquad \begin{pmatrix} 0 \\ \frac{\epsilon_2}{R_2} \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ \frac{R_1}{R_2} & 1 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{11}$$

Now we can subtract (just like plugging in) the second equation to the first, to get rid of the I_2 in the first equation in terms of I_1.

$$\begin{array}{ccccccc} \frac{-\epsilon_2}{R_2} &=& -\left(\frac{R_1}{R_2}+1\right) I_1 &+& 0 &+& I_3 \\ \frac{\epsilon_2}{R_2} &=& \frac{R_1}{R_2} I_1 &+& I_2 &+& 0 \\ \epsilon_3 &=& R_1 I_1 &+& 0 &+& R_3 I_3 \end{array} \qquad \begin{pmatrix} \frac{-\epsilon_2}{R_2} \\ \frac{\epsilon_2}{R_2} \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{R_2}-1 & 0 & 1 \\ \frac{R_1}{R_2} & 1 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{12}$$

Now lets solve the last equation for I_3 by dividing by R_3...

$$\begin{array}{ccccccc} \frac{-\epsilon_2}{R_2} &=& -\left(\frac{R_1}{R_2}+1\right) I_1 &+& 0 &+& I_3 \\ \frac{\epsilon_2}{R_2} &=& \frac{R_1}{R_2} I_1 &+& I_2 &+& 0 \\ \frac{\epsilon_3}{R_3} &=& \frac{R_1}{R_3} I_1 &+& 0 &+& I_3 \end{array} \qquad \begin{pmatrix} \frac{-\epsilon_2}{R_2} \\ \frac{\epsilon_2}{R_2} \\ \frac{\epsilon_3}{R_3} \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{R_2}-1 & 0 & 1 \\ \frac{R_1}{R_2} & 1 & 0 \\ \frac{R_1}{R_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{13}$$

...and subtracting the last equation to the first, to get rid of the I_3 in the first equation in terms of I_1.

$$\begin{array}{ccccccc} \frac{-\epsilon_3}{R_3}-\frac{\epsilon_2}{R_2} &=& -\left(\frac{R_1}{R_3}+\frac{R_1}{R_2}+1\right) I_1 &+& 0 &+& 0 \\ \frac{\epsilon_2}{R_2} &=& \frac{R_1}{R_2} I_1 &+& I_2 &+& 0 \\ \frac{\epsilon_3}{R_3} &=& \frac{R_1}{R_3} I_1 &+& 0 &+& I_3 \end{array} \qquad \begin{pmatrix} \frac{-\epsilon_3}{R_3}-\frac{\epsilon_2}{R_2} \\ \frac{\epsilon_2}{R_2} \\ \frac{\epsilon_3}{R_3} \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{R_3}-\frac{R_1}{R_2}-1 & 0 & 0 \\ \frac{R_1}{R_2} & 1 & 0 \\ \frac{R_1}{R_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{14}$$

Now we have an equation with just I_1 and the R_i and ϵ_i that were given. Dividing through by the junk in front of I_1 we have

$$\begin{array}{ccccccc} \frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{R_1}{R_3}+\frac{R_1}{R_2}+1} &=& I_1 &+& 0 &+& 0 \\ \frac{\epsilon_2}{R_2} &=& \frac{R_1}{R_2} I_1 &+& I_2 &+& 0 \\ \frac{\epsilon_3}{R_3} &=& \frac{R_1}{R_3} I_1 &+& 0 &+& I_3 \end{array} \qquad \begin{pmatrix} \frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{R_1}{R_3}+\frac{R_1}{R_2}+1} \\ \frac{\epsilon_2}{R_2} \\ \frac{\epsilon_3}{R_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{R_1}{R_2} & 1 & 0 \\ \frac{R_1}{R_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{15}$$

A solution! We know have an explicit expression for I_1. Going back and subtracting R_1/R_2 time the first equation from the middle lets us solve for I_2.

$$\begin{array}{ccccccc} \frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{R_1}{R_3}+\frac{R_1}{R_2}+1} &=& I_1 &+& 0 &+& 0 \\ \frac{\epsilon_2}{R_2}-\frac{1}{R_2}\frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{1}{R_3}+\frac{1}{R_2}+\frac{1}{R_1}} &=& 0 &+& I_2 &+& 0 \\ \frac{\epsilon_3}{R_3} &=& \frac{R_1}{R_3} I_1 &+& 0 &+& I_3 \end{array} \qquad \begin{pmatrix} \frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{R_1}{R_3}+\frac{R_1}{R_2}+1} \\ \frac{\epsilon_2}{R_2}-\frac{1}{R_2}\frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{1}{R_3}+\frac{1}{R_2}+\frac{1}{R_1}} \\ \frac{\epsilon_3}{R_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{R_1}{R_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{16}$$

And finally subtracting R_1/R_3 time the first equation from the last lets us solve for I_3.

$$\begin{array}{ccccccc} \frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{R_1}{R_3}+\frac{R_1}{R_2}+1} &=& I_1 &+& 0 &+& 0 \\ \frac{\epsilon_2}{R_2}-\frac{1}{R_2}\frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{1}{R_3}+\frac{1}{R_2}+\frac{1}{R_1}} &=& 0 &+& I_2 &+& 0 \\ \frac{\epsilon_3}{R_3}-\frac{1}{R_3}\frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{1}{R_3}+\frac{1}{R_2}+\frac{1}{R_1}} &=& 0 &+& 0 &+& I_3 \end{array} \qquad \begin{pmatrix} \frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{R_1}{R_3}+\frac{R_1}{R_2}+1} \\ \frac{\epsilon_2}{R_2}-\frac{1}{R_2}\frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{1}{R_3}+\frac{1}{R_2}+\frac{1}{R_1}} \\ \frac{\epsilon_3}{R_3}-\frac{1}{R_3}\frac{\frac{\epsilon_3}{R_3}+\frac{\epsilon_2}{R_2}}{\frac{1}{R_3}+\frac{1}{R_2}+\frac{1}{R_1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{17}$$

Yuck! Why would anyone want to solve equations like that? You have to do all the same work in the matrix version, but you're rewriting everything at each step! What gives? The power of linear algebra is that, while it is long and tedious for us (see above), it is very easy for a computer (see below). Understanding why means taking a step back and getting an abstract view of the solution we just worked through.

3 Solving matrix equations with a computer

When we started, we had an equation like this

$$[\epsilon] = [R]\,[I] \tag{18}$$

Wait a second, the ϵs were voltages, so that looks a lot like plain old $V = IR$! We know how to get the current then, it's just

$$R^{-1}V = I\ . \tag{19}$$

Maybe we can think about our matrix equation like this, we just need to figure out what $[R]^{-1}$ means. What is R^{-1} anyway? It's the *inverse* of R; the thing that, when multiplied by R, gives one. $[R]^{-1}$ is also just the inverse of $[R]$, so $[R]^{-1} \cdot [R] = 1$.

Inverting a matrix by hand is basically what we were doing in our solution above, but if we can get our computer (or calculator) to find $[R]^{-1}$ for us, we don't have to do any of the messy algebra. We can just get our solution via

$$[R]^{-1}[\epsilon] = [I] \tag{20}$$

On the TI-83+, that's pretty much all there is to it. You can enter your $[R]$ and $[\epsilon]$ matrices in the `[2nd] MATRX` $\rightarrow$ `EDIT` menu (calling them $[A]$ and $[B]$. Then just type out

$$[A]^{-1} * B \tag{21}$$

using the `[2nd] MATRX` $\rightarrow$ `NAMES` menu to generate the $[A]$ and $[B]$ symbols.

On the TI-89, you can enter the matrices straight from the command line, using comas to separate the columns, and semi-colons to separate the rows. Using the numbers from Problem 35 that gives

$$[-1, 1, 1; 8, 6, 0; 8, 0, 4] \rightarrow A \tag{22}$$

$$[0; 4; 12] \rightarrow I \tag{23}$$

$$A^{-1} * I \tag{24}$$

$$\begin{pmatrix} 0.8462 \\ -0.4615 \\ 1.3077 \end{pmatrix} \tag{25}$$

(I don't have a TI-89, so if this is wrong, let me know...).