Recitation 6

Chapter 21

Problem 35. *Determine the current in each branch of the circuit shown in Figure P21.35.*

Let I_1 be the current on the left branch (going down), I_2 be the current on the middle branch (going up), and I_3 be the current on the right branch (going up). From Kirchoff's junction rule, we know.

$$I_1 = I_2 + I_3 \tag{1}$$

Let $\varepsilon_2 = 4.00$ V be the voltage across the middle battery, and $\varepsilon_3 = 12.0$ V be the voltage across the right battery.
Using our knowledge of parallel resistors, we find

$$R_1 = 8.00\Omega \tag{2}$$
$$R_2 = 5.00\Omega + 1.00\Omega = 6.00\Omega \tag{3}$$
$$R_3 = 3.00\Omega + 1.00\Omega = 4.00\Omega \tag{4}$$

We can use Ohm's law to find the voltage drops across them in the direction of their current.
Now using Kirchoff's loop rule on the left-center and left-right loops respectively we have

$$0 = \varepsilon_2 - I_2R_2 - I_1R_1 \tag{5}$$
$$0 = \varepsilon_3 - I_3R_3 - I_1R_1 \tag{6}$$

So we have our three equations relating our unknown currents. If you're comfortable with linear algebra, you can express these as a matrix

$$\begin{pmatrix} 0 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ R_1 & R_2 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{7}$$

Inverting the 3x3 matrix, we get

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -0.2308 & 0.0385 & 0.0577 \\ 0.3077 & 0.1154 & -0.0769 \\ 0.4615 & -0.0769 & 0.1346 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 4.00 \\ 12.0 \end{pmatrix} = \begin{pmatrix} 0.8462 \\ -0.4615 \\ 1.3077 \end{pmatrix} \text{ A} \tag{8}$$

Where $I_2 < 0$ indicates that current actually flows in the opposite direction to what we expected.
If you're not comfortable with linear algebra, you can solve the equations using your method of choice. If no methods make sense to you, come talk to me or get someone else to teach you one.

Problem 38. *The following equations describe an electric circuit:*

$$-(220\Omega)I_1 + 5.80\ V - (370\Omega)I_2 = 0 \tag{9}$$
$$(370\Omega)I_2 + (150\Omega)I_3 - 3.10\ V = 0 \tag{10}$$
$$I_1 + I_3 - I_2 = 0 \tag{11}$$

(a) Draw a diagram of the circuit. (b) Calculate the unknowns and identify the physical meaning of each unknown.

(b) Solve using your method of choice. With linear algebra:

$$\begin{pmatrix} 5.80\text{ V} \\ 3.10\text{ V} \\ 0 \end{pmatrix} = \begin{pmatrix} 220\Omega & 370\Omega & 0 \\ 0 & 370\Omega & 150\Omega \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{12}$$

Inverting the 3x3 matrix, we get

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0.0031 & -0.0022 & 0.3267 \\ 0.0009 & 0.0013 & -0.1942 \\ -0.0022 & 0.0035 & 0.4791 \end{pmatrix}^{-1} \begin{pmatrix} 5.80\text{ V} \\ 3.10\text{ V} \\ 0 \end{pmatrix} = \begin{pmatrix} 11.0 \\ 9.13 \\ -1.87 \end{pmatrix} \text{ mA} \tag{13}$$

Problem 42. *A $C = 2.00$ nF capacitor with an intial charge of $Q = 5.10\ \mu C$ is discharged through an $R = 1.30\Omega$ resistor. (a) Calculate the current in the resistor $t_a = 9.00\ \mu s$ after the resistor is connected across the terminals of the capacitor. (b) What charge remains on the capacitor after $t_b = 8.00\ \mu s$? (c) What is the maximum current in the resistor?*

(a) The current through the entire circuit follows

$$I = \frac{Q}{RC}e^{-t/RC} \tag{14}$$

So

$$I(t_a) = \frac{5.10 \cdot 10^{-6}\text{ C}}{1.30\Omega \cdot 2.00 \cdot 10^{-6}\text{ F}} e^{\frac{-9.00 \cdot 10^{-6}\text{ s}}{1.30\Omega \cdot 2.00 \cdot 10^{-6}\text{ F}}} = 61.6\text{ mA} \tag{15}$$

(b) The charge on the capacitor follows

$$q = Qe^{-t/RC} \tag{16}$$

So

$$I(t_a) = 5.10 \cdot 10^{-6}\text{ C}e^{\frac{-8.00 \cdot 10^{-6}\text{ s}}{1.30\Omega \cdot 2.00 \cdot 10^{-6}\text{ F}}} = 235\ \mu\text{C} \tag{17}$$

(c) Plugging $t = 0$ into our equation from (a) we have

$$I_{max} = \frac{Q}{RC} = 1.96\text{ A} \tag{18}$$

Problem 45. *The circuit in Figure P21.45 has been connected for a long time. (a) What is the voltage V_c across the capacitor? (b) If the battery is disconnected, how long does it take the capacitor to discharge to $V_c' = 1/10 \cdot V$?*

Labeling the resistors counterclockwise from the upper left we have $R_1 = 1.00\Omega$, $R_2 = 4.00\Omega$, $R_3 = 2.00\Omega$, and $R_4 = 8.00\Omega$. Let $V = 10.0$ V be the voltage on the battery and $C = 1.00\ \mu$F be the capacitance of the capacitor.

(a) Because the system has been running for a long time, the system must be close to equilibrium. Therefore, the current through the capacitor must be zero (otherwise the voltage across the capacitor would be changing, and you wouldn't be at equilibrium). The resistor bridge then reduces to two parallel circuits, and we can apply Ohm's law to determine V_c

Starting with the left side of the bridge (calling the current I_L),

$$V = I_L(R_1 + R_2) \tag{19}$$

$$I_L = \frac{V}{R_1 + R_2} \tag{20}$$

And on the right calling the current I_R

$$I_R = \frac{V}{R_3 + R_4} \tag{21}$$

So using Ohm's law to compute the voltage across the capacitor, we call the voltage on the bottom wire 0 and have the voltage V_L on the left at

$$V_L = I_L R_2 = \frac{V R_2}{R_1 + R_2} = 8\text{ V} \tag{22}$$

And the voltage V_R to the right of the capacitor is

$$V_R = I_R R_3 = \frac{V R_3}{R_3 + R_4} = 2\text{ V} \tag{23}$$

So the voltage across the capacitor is

$$V_c = V_L - V_R = 6\text{ V} \tag{24}$$

(b) Once we remove the battery, we see that the capacitor discharges through two paths in parallel, $R_1 \to R_4$ and $R_2 \to R_3$. The eqivalent resistances of these two parallel branches (top and bottom) are

$$R_T = R_1 + R_4 \tag{25}$$

$$R_B = R_2 + R_3 \tag{26}$$

So the total equivalent resistance is

$$R = \left(\frac{1}{R_T} + \frac{1}{R_B}\right)^{-1} = 3.60\Omega \tag{27}$$

The voltage of a discharging capacitor depends on time according to

$$V_c' = V_c e^{-t/RC} \tag{28}$$

So using $V_c' = V_c/10$ we have

$$10 = \frac{V_c}{V_c'} = \frac{V_c}{V_c e^{-t/RC}} = e^{t/RC} \tag{29}$$

$$\ln(10) = \frac{t}{RC} \tag{30}$$

$$t = RC\ln(10) = 8.29\ \mu\text{s} \tag{31}$$

Problem 53. *An electric heater is rated at $P_H = 1500$ W, a toaster at $P_T = 750$ W, and an electric grill at $P_G = 1000$ W. The three appliances are connected to a common $V = 120$ V household circuit. (a) How much current does each draw? (b) Is a circuit with a $V_{max} = 25.0$ A circuit breaker sufficient in this situation? Explain your answer.*

(a) Using $P = IV$ we have

$$I_H = \frac{P_H}{V} = 12.5 \text{ A} \tag{32}$$
$$I_T = \frac{P_T}{V} = 6.25 \text{ A} \tag{33}$$
$$I_G = \frac{P_G}{V} = 8.33 \text{ A} \tag{34}$$
$$\tag{35}$$

(b) If all the appliances are running together, the circuit draws

$$I = I_H + I_T + I_G = 27.1 \text{ A} \tag{36}$$

So you will be fine with a 25 A breaker unless you plan to run all three at the same time.

Problem 58. *A battery with emf ε is used to charge a capacitor C through a resistor R as shown in Figure 21.25. Show that half the energy supplied by the battery appears as internal energy in the resistor and that half is stored in the capacitor.*

The total current through the system is given by

$$I = I_o e^{-t/RC} = \frac{\varepsilon}{R} e^{-t/RC} \tag{37}$$

Which allows us to compute the energy put out by the battery. Power is the time derivative of energy so

$$E_b = \int_0^\infty P \cdot dt \tag{38}$$
$$= \int_0^\infty I\varepsilon \cdot dt \tag{39}$$
$$= \frac{\varepsilon^2}{R} \int_0^\infty e^{-t/RC} \cdot dt \tag{40}$$
$$= \frac{\varepsilon^2}{R} -RCe^{-t/RC}\Big|_0^\infty \tag{41}$$
$$= \frac{\varepsilon^2}{R} \left(0 - (-RCe^0)\right) \tag{42}$$
$$= C\varepsilon^2 \tag{43}$$

Similarly for the energy absorbed by the resistor

$$E_r = \int_0^\infty P \cdot dt \tag{44}$$
$$= \int_0^\infty I^2 R\varepsilon \cdot dt \tag{45}$$
$$= \frac{\varepsilon^2}{R} \int_0^\infty e^{-2t/RC} \cdot dt \tag{46}$$
$$= \frac{\varepsilon^2}{R} \frac{-RC}{2} e^{-2t/RC}\Big|_0^\infty \tag{47}$$
$$= \frac{E_b}{2} = \frac{1}{2} C\varepsilon^2 \tag{48}$$

And we already know the energy stored in a capacitor with a voltage ε is

$$E_c = \frac{1}{2} C\varepsilon^2 = \frac{E_b}{2} \tag{49}$$

So the battery energy splits evenly between the capacitor and the resistor, and we're done.