Useful Equations

Introduction

I've just gone through and compiled the important equations from the ends of the various chapters, and the equations given on the quizzes so far. Hope this helps :).

Chapter 1: Introduction and vectors

Trig

sohcahtoa

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \tag{1}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{2}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \tag{3}$$

Vectors

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}} \tag{4}$$

Chapter 2: Motion in one dimension

Definitions

Average quantities

$$\mathbf{v}_{avg} \equiv \frac{\Delta \mathbf{x}}{\Delta t} \tag{5}$$

$$\mathbf{a}_{avg} \equiv \frac{\Delta \mathbf{v}}{\Delta t} \tag{6}$$

Instantaneous quantities

$$\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \tag{7}$$

$$\mathbf{a} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{8}$$

Equations of motion

For constant velocity problems (integrating the instantaneous velocity definition)

$$x_f = x_i + v_x t \tag{9}$$

And for constant acceleration problems (integrating the instantaneous acceleration definition twice, and manipulating a bit).

$$v_{xf} = v_{xi} + a_x t \tag{10}$$

$$x_f = x_i + \frac{1}{2}(v_{xi} + v_{xf})t \tag{11}$$

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t \tag{12}$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i) \tag{13}$$

Where eqn. 13 comes from solving eqn 12 for t using the quadratic formula and plugging the result into eqn 10.

The quadratic formula says that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14}$$

Yeilds two values of x that solve the quadratic equation

$$0 = ax^2 + bx + c \tag{15}$$

Chapter 3: Motion in two dimensions

Constant acceleration

Applying our 1D equations of motion to each direction, we get the multidimensional formulas:

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \tag{16}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t \tag{17}$$

In the special case of projectile motion with $a_x = 0$ and $a_y = g$ these reduce to

$$v_{xf} = v_{xi} = \text{constant} \tag{18}$$

$$x_f = x_i + v_x t \tag{19}$$

$$v_{yf} = v_{yi} - gt \tag{20}$$

$$y_f = y_i + v_{yi} - \frac{1}{2}gt^2 \tag{21}$$

We could apply our accelerations to all four of the constant 1D acceleration equations in chapter 2, but you get the idea...

Circular motion

A particle moving in a circle of radius r with velocity v has a centerward acceleration of

$$a_c = \frac{v^2}{r} \tag{22}$$

Frames of reference

If an observer O' is moving with velocity $\mathbf{v}_{O'O}$ with respect to observer O, their measurements of the velocity of a particle located at point P are related according to

$$\mathbf{v}_{PO} = \mathbf{v}_{PO'} + \mathbf{v}_{OO'} \tag{23}$$

Chapter 4: The laws of motion

Newton's laws

1. An object in motion will remain in motion unless acted upon by an outside force.
2. $\sum \mathbf{F} = m\mathbf{a}$
3. For every force there is an equal and opposite reaction force.

Note that the two forces referenced in the 3rd law belong to two *different* free body diagrams (FBDs). For example, the sun and earth attract each other gravitationally. Let F be the magnitude of the force, and $\hat{\mathbf{i}}$ be the direction from the earth to the sun. The force on the earth due to the sun (showing up on the earth's FBD) is $F\hat{\mathbf{i}}$ and the force on the sun due to the earth (showing up on the sun's FBD) is $-F\hat{\mathbf{i}}$.

Chapter 5: More applications of Newton's laws

Friction

Let μ_s and μ_k be the static and kinetic coefficients of friction (respectively) between an object and a surface, and let F_N be the normal force on the object due to the surface. The respective forces of friction are given by

$$F_{sf} \le \mu_s F_N \tag{24}$$

$$F_{kf} = \mu_k F_N \tag{25}$$

Drag

Objects moving through viscous materials (air, water, etc.) experience a velocity dependent resistive force

$$F_{drag} = -bv \tag{26}$$

Where b depends on the particular system under consideration.

Chapter 6: Energy and energy transfer

Work

In general

$$W \equiv \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{27}$$

Or for the one dimensional case

$$W \equiv \int_{x_i}^{x_f} \mathbf{F} \cdot \hat{\mathbf{i}} \mathrm{d}x \tag{28}$$

Where the dot product $\mathbf{A} \cdot \mathbf{B}$ is defined as

$$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}||\mathbf{B}| \cos\theta \tag{29}$$

Where θ is the angle between the vectors. So in the 1D, constant force-and-angle case

$$W = \mathbf{F} \cdot \Delta\mathbf{r} \tag{30}$$

The force from a spring is given by Hooke's law

$$F_s = -kx \tag{31}$$

So the work done by a spring from x_i to x_f is

$$W_s = \int_{x_i}^{x_f} (-kx)\mathrm{d}x = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 \tag{32}$$

Kinetic energy

$$K = \frac{1}{2}mv^2 \tag{33}$$

Work-kinetic energy theorem

$$W_{net} = K_f - K_i = \Delta K \tag{34}$$

Power

$$P_{avg} \equiv \frac{W}{\Delta t} \tag{35}$$

$$P \equiv \frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v} = \frac{\mathrm{d}E}{\mathrm{d}t} \tag{36}$$

Chapter 7: Potential energy

Conservative forces

A force is *conservative* if the work it does on a particle is independent of the path the particle takes between two given points. The potential energy change is the inverse of the work done by the force

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x dx \tag{37}$$

Or, taking the derivative of both sides with respect to x

$$F_x = -\frac{\mathrm{d}U}{\mathrm{d}x} \tag{38}$$

The gravitational potential of a particle under a gravitational force $F_g = mg$ is (relative to the energy at some height $h_r = 0$)

$$U_g = -\int_0^h (-mg)\mathrm{d}y = mgh \tag{39}$$

The spring potential of a particle under a spring force $F_s = -kx$ is (relative to the unstretched energy at $x_r = 0$)

$$U_s = -\int_0^x (-kx)\mathrm{d}x = \frac{1}{2}kx^2 \tag{40}$$

The gravitational potential energy of a particle under Newton's gravitational force F_G is (relative to the energy at $r_r = \infty$)

$$U_G = -\int_\infty^r \left(\frac{-Gm_1m_2}{r^2}\right)\mathrm{d}r = Gm_1m_2\int_\infty^r \frac{1}{r^2}\mathrm{d}r = \left.\frac{-Gm_1m_2}{r}\right|_\infty^r = \frac{-Gm_1m_2}{r} \tag{41}$$

And so on for any other conservative forces...

Mechanical energy

The total mechanical energy at any moment is

$$E_{mech} \equiv K + U \tag{42}$$

Conserving energy using this formula

$$E_i = K_i + U_i = E_f = K_f + U_f \tag{43}$$
$$\Delta K = -\Delta U \tag{44}$$

Which is identical to the work-kinetic energy theorem from Chapter 6 (eqn 34 with $W_{net} = -\Delta U$). The only difference between these two approaches is that nonconservative forces do not have really well defined "potential energies".

Equilibria

In a potential energy diagram, a point of *stable equilibrium* is a local minimum, a point of *unstable equilibrium* is a local maximum, and a region of *neutral equilibrium* is a region of constant potential energy.

Chapter 8: Momentum and collisions

Linear momentum

Momentum

$$\mathbf{p} \equiv m\mathbf{v} \tag{45}$$

is conserved

$$\sum \mathbf{p}_i = \sum \mathbf{p}_f \tag{46}$$

Impulse

The impulse-momentum theorem

$$\mathbf{I} = \int_{t_1}^{t_2} \sum \mathbf{F} \mathrm{d}t = \Delta \mathbf{p} \tag{47}$$

Types of collisions

- Inelastic collision: kinetic energy is not conserved.
- Perfectly inelastic collision: the particles stick together afterwards.
- Elastic collision: kinetic energy is conserved.

In all types of collisions (without external forces), momentum is conserved.

Center of mass

$$\mathbf{r}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \tag{48}$$

Chapter 10: Rotational motion

Definitions

$$\omega \equiv \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{49}$$
$$\alpha \equiv \frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{50}$$

Relations to linear quantities

(All of the linear quanties are tangential)

$$s_t = r\theta \tag{51}$$
$$v_t = r\omega \tag{52}$$
$$a_t = r\alpha \tag{53}$$

For constant α problems, all of the constant acceleration equations from Chapter 2 are still valid with the proper substitutions.

Analogs of other linear properties

The rotational motion equivalent of mass is the *moment of inertia*

$$I = \sum_i m_i r_i^2 \tag{54}$$
$$I = \int \rho r^2 \mathrm{d}V \tag{55}$$

And the moments of inertia for some common shapes (Table 10.2, page 300)

Hoop/ring about axis	MR^2
Solid cylinder about axis	$\frac{1}{2}MR^2$
Hollow cylinder about axis	$\frac{1}{2}M(R_i^2 + R_o^2)$
Long thin rod perp to axis through center	$\frac{1}{12}ML^2$
Long thin rod perp to axis through end	$\frac{1}{3}ML^2$
Solid sphere about diameter	$\frac{2}{5}MR^2$
Thin spherical shell about diameter	$\frac{2}{3}MR^2$
Rectangular plate perp to one side through center	$\frac{1}{12}M(a^2 + b^2)$

The kinetic energy of a rotating body is given by

$$K = \frac{1}{2}I\omega^2 \tag{56}$$

(Note that the kinetic energy of any given particle should be expressed as *either* a linear or a rotational kinetic energy, not as *both* at once. You *can* use linear kinetic energy for one particle and rotational kinetic energy for another to find the kinetic energy at a single point in time.)

The rotational equivalent to force is torque

$$\tau \equiv \mathbf{r} \times \mathbf{F} \tag{57}$$

Where the cross product between two vectors defined by

$$\mathbf{A} \times \mathbf{B} \equiv |\mathbf{A}||\mathbf{B}| \sin\theta \widehat{rhr} \tag{58}$$

Where $\widehat{rhr}$ is a unit vector in the direction specified by the right-hand rule.

The analog to Newton's second law is

$$\sum \tau = I\alpha = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{59}$$

The angular momentum $\mathbf{L}$ is given by

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = I\omega \tag{60}$$

Angular momentum is conserved if there are no external torques on the system. As with rotational kinetic energy, you should be careful to avoid double counting by using both rotational and linear momentum to compute the momentum of the same particle, although here the units do not match (another reason to keep track of your units!).

Quiz 1: Vector addition and projectile motion

$$v_2^2 = v_1^2 + 2a(x_2 - x_1) \qquad \text{(From 13)} \tag{61}$$
$$v_2 = v_1 + at \qquad \text{(From 10)} \tag{62}$$
$$x_2 = x_1 + v_1 t + \frac{1}{2}at^2 \qquad \text{(From 12)} \tag{63}$$

Quiz 2: Projectile motion, Newton's laws, and reference frames

$$y = y_0 + v_0(\sin\theta)t - \frac{1}{2}gt^2 \quad \text{(From 21)} \tag{64}$$
$$x = v_0(\cos\theta)t \quad \text{(From 19)} \tag{65}$$
$$v_y = v_0(\sin\theta) - gt \quad \text{(From 20)} \tag{66}$$
$$v_x^2 = v_{x0}^2 + 2a_x\Delta x \quad \text{(From 13)} \tag{67}$$
$$\sum F_{ext} = \frac{\mathrm{d}p}{\mathrm{d}t} \quad \text{(From Newton's 2nd law and 45)} \tag{68}$$
$$\sum F_{ext} = ma \quad \text{(Newton's 2nd law)} \tag{69}$$
$$R = (v_0^2 \sin 2\theta)/g \quad \text{(Range eqn 3.16 from book page 76)} \tag{70}$$
$$\mathbf{V}_{12} = \mathbf{V}_{1G} + \mathbf{V}_{G2} \quad \text{(From 23)} \tag{71}$$

Where they have also used sohcahtoa to get $v_{xi} = v_0 \cos\theta$ and $v_{yi} = v_0 \sin\theta$.

Quiz 3: Friction, momentum, work, kinetic energy, and springs

$$\Delta W = \Delta KE \quad \text{(From 34)} \tag{72}$$
$$KE = 1/2mv^2 \quad \text{(From 33)} \tag{73}$$
$$\Delta W_{ext} = 1/2kx^2 \quad \text{(From 32 with } x_f = 0\text{)} \tag{74}$$
$$F = -kx \quad \text{(From 31)} \tag{75}$$
$$\Delta W_{12} = \int_1^2 F.dx \quad \text{(From 28)} \tag{76}$$
$$\mathbf{A}.\mathbf{B} = AB\cos\theta \quad \text{(From 29)} \tag{77}$$
$$\Delta W_{12} = \mathbf{F}.\mathbf{X} \quad \text{(From 30)} \tag{78}$$