HONORS 301: SPECIAL RELATIVITY
Winter 2008-2009
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Lecture Notes #4: Lorentz Transformation
Thursday, January 29, 2009

0 Preliminaries

Required Reading: *Spacetime Physics*, ch. L (between chapters 3 and 4, like the extra floor in the movie "Being John Malkovich")

Homework: HW3 is due today! No homework due next week because...

Midterm: In class on Thursday, February 5.

1 The Lorentz Transformation

Using only the invariance of the spacetime interval, we demonstrated

- Relativity of Simultaneity
- Time Dilation
- Length Contraction

We also inferred from pure logic

- Invariance of transverse dimension
- Transverse simultaneity

If we think of all events as specified by their spacetime coordinates, then we can derive a set of equations that allows us to transform observations from one inertial frame to any other inertial frame. So, let's start with the space and time measurements from the rods and clocks in an inertial frame, say, inside a rocket and apply this transformation to compute what would be observed in the laboratory frame.

Today we'll derive the Lorentz Transformation and apply it. Among other things, we'll look at how velocities combine in Special Relativity and show that you can't get faster than the speed of light. We'll also look at what would happen if you did travel faster than the speed of light and what utter nonsense that would entail.

IMPORTANT: we are going to work in units where $c = 1$. Therefore, all velocities are dimensionless, just the fraction of the speed of light. For example, $v = 0.8$ means four-fifths of the speed of light. Time and distance will be in the same units, assuming that we already converted time into distance units or vice versa.

2 Coordinates in different frames

Start by laying down a three dimensional grid of rods in the rocket frame. There are 3 spatial dimensions. Label them x', y', z'. The time coordinate recorded by the clocks in this frame we'll label t'. Also lay down a three dimensional grid of rods and clocks in the laboratory frame, with spatial dimension x, y, z, and clock time t. Be careful to align x with x', y with y', and z with z'. To make things easy, we'll have the rocket frame move relative to the laboratory frame so that all the motion is in the positive x direction with relative velocity v_{rel}. Note that there's nothing special about the x direction; we could always relabel our coordinates.

We'll also set the time coordinates so that the zeros of both space and time for both frames are set by the same reference event. For example, set off a flash in the rocket frame at some location. Call that point $x' = y' = z' = t' = 0$. Wherever that flash occurs in the lab frame at the same time call $x = y = z = t = 0$.

3 Properties of the Lorentz Transformation

Suppose that we know x', y', z', and t' for an event in the rocket frame. Can we write down the set of equations that tells us x, y, z, and t?

Before we start, let's think about what properties these transformation equations must have:

- Obviously, the equations must be consistent with invariance of the spacetime interval – we'll use this invariance to derive the equations.
- Speed of light must be the maximum speed allowed.

- What will the transformation depend on? Relative velocity of the free-fall frames! That's the only variable allowed. The laws of physics are the same in all free-falling frames. The only difference between these frames is their relative velocity.

- Transformation must be INVERTIBLE. What do you mean invertible? That means that if we transform the coordinates of an event from the rocket frame to the lab frame and then apply the transformation again to get from the lab frame to the rocket frame, we had better get back to where we started!

- Transformation must be LINEAR. That means that the coordinates in one frame depend on the first power of the coordinates in another frame. So, we'll have equations of the form $x = Ax' + By' + Cz' + Dt'$ but not $x = Ax'^2 + By'^3 + Cz'^5 + D\sqrt{t'}$ or anything like that. We'll come back to this in just a minute...

4 Derivation for x'=0

First, we can work out the really easy part. If the relative motion is in the x direction, what are the y and z coordinates of an event in the rocket frame? These are transverse to the direction of motion, so by the invariance of transverse dimensions, we immediately see that $y = y'$, and $z = z'$. Not sure about this? Look at the grid of rods and clocks! The y and z axes in the rocket and lab frames are always aligned.

What about x and t? First consider a really simple case. We set the zeros of the lab and rocket frames with some event that we said occured at $\{0, 0, 0, 0\}$ in both frames. Now have a second event at $x' = 0$ and t' in the rocket frame, say, a flash from a light bulb.

Given $x' = 0$ and t' what are x and t? Easy – use the invariance of the spacetime interval. The interval is, of course, the interval between the zero of the spacetime coordinates and the event, in this case the flash. Where is this flash in the lab frame? The rocket moves at v_{rel} in the lab frame, so the location $x' = 0$ moves to $x = v_{rel}t$ in time t. Now use the invariance of the spacetime interval:

$$(t')^2 - (x')^2 = t^2 - x^2$$

Substitute in $x' = 0$ and $x = v_{rel}t$

$$(t')^2 = t^2 - (v_{rel}t)^2 = t^2(1 - v_{rel}^2)^2$$

or simply

$$t = \frac{t'}{\sqrt{1 - v_{rel}^2}}$$

This, of course, is the equation for time dilation. A shorthand that is often used is to define the "Lorentz factor",

$$\gamma \equiv \frac{1}{\sqrt{1 - v_{rel}^2}}$$

thus

$$t = \gamma t'.$$

Note that the Lorentz factor is always greater than or equal to 1, so it's clear that there is time *dilation*. From before, $x = v_{rel}t$, thus

$$x = v_{rel}\gamma t'.$$

We're done! But only for the case where all the events occur at $x' = 0$ in the rocket frame. Can we generalize this to work for any location in any freely-falling frame?

5 Derivation for more general case

To work out the more general case, we'll use three things:

- Transformation must be linear
- Transformation must yield the preceding equation for $x' = 0$ case
- Invariance of spacetime interval

5.1 Linearity of transformation

Why does the transformation have to be linear? Because otherwise it would depend on the zero of the coordinate system, the arbitrary spacetime location that you labelled $\{0, 0, 0, 0\}$. Take a closer look at the transformation between rocket and lab frame for events that occur at $x' = 0$ in the rocket frame. Suppose that there are three different flashes at $x' = 0$, at different times $t'_1 = 0, t'_2 = 1, t'_3 = 2$. If the relative velocity of the frames is such that $\gamma = 2$, then using $t = \gamma t'$, $t_1 = 0, t_2 = 2, t_3 = 4$. The size of the intervals between the first and second and second and third events are the same, just stretched due to time dilation.

Now imagine any other dependence of t on t'. For example, $t = \gamma t'^3$. Then $t_1 = 0, t_2 = 2, t_3 = 16$. The intervals are now very different, but this is ridiculous – the time interval between events should not depend on where we set the zero of the coordinate system!

THIS IS AS FAR AS WE ACTUALLY GOT ON THURSDAY, JANUARY 29, SO THE REST WILL NOT BE ON THE MIDTERM.

So, let's demand that the relationship between coordinates be linear:

$$\begin{aligned} t &= Bx' + Dt' \\ x &= Gx' + Ht' \end{aligned}$$

where the constants B, D, G, H are things we have to find. The letters are chosen just to avoid confusion with other symbols, like c.

We know what this looks like for $x' = 0$,

$$\begin{aligned} t &= \gamma t' \\ x &= v_{rel}\gamma t' \end{aligned}$$

Thus

$$\begin{aligned} t &= Bx' + \gamma t' \\ x &= Gx' + v_{rel}\gamma t' \end{aligned}$$

Now let's use invariance of the spacetime interval to work out the rest:

$$t^2 - x^2 = t'^2 - x'^2$$

Plug in from above

$$(Bx' + \gamma t')^2 - (Gx' + v_{rel}\gamma t')^2 = t'^2 - x'^2$$

Next, multiply out the squared terms, then group together the terms that have t'^2, x'^2 and $x't'$

We won't go through all the algebra in class. You can follow this in your book on pages 101-102. Let's look at the solution:

$$\begin{aligned} t &= v_{rel}\gamma x' + \gamma t' \\ x &= \gamma x' + v_{rel}\gamma t' \\ y &= y' \\ z &= z' \end{aligned}$$

where velocity is in units of light speed (so $c = 1$). And note CAREFULLY how this was derived: $\{t, x, y, z\}$ are the coordinates in the lab frame. The primed quantities $\{t', x', y', z'\}$ are the coordinates in the rocket frame. The velocity v_{rel} is the velocity of the rocket as measured by the observer in the lab frame. We oriented our axes so that the relative motion is in the x direction. Remember, there's nothing magic about the "rocket" frame or the "lab" frame. Here's one way to think about it: in these equations, the unprimed quantities and the relative velocity are what YOU measure. The primed quantities are measured in the frame that appears to you to be moving.

5.2 Invertibility of transformation

Remember we said that this transformation had to be invertible. Let's check that this works! What's with the primed and uprimed frame? Just the sign of the velocity. We defined things so that the lab observer sees the rocket move in the positive x direction. Therefore, the rocket observer sees the lab move in the negative x direction, right? So all we need to do to get the inverse transformation is to swap the primed for unprimed, and vice versa, and change the sign of the velocity. Note carefully that γ depends on the square of the velocity, so direction does not matter. This had better work – otherwise the laws of physics are different!

Thus we get simply:

$$\begin{aligned} t' &= -v_{rel}\gamma x + \gamma t \\ x' &= \gamma x - v_{rel}\gamma t \\ y' &= y \\ z' &= z \end{aligned}$$

Do a quick example to convince you: Sample Problem L-1 on p. 104.

6 Addition of velocities

Using the Lorentz Transformation, we can derive an equation that shows how velocities combine in Special Relativity. Obviously, they don't simply add up, otherwise we could get speeds greater than the speed of light. Imagine a frame moving at $0.5c$ relative to us that fires a projectile at $0.6c$. Does the projectile move at $1.1c$ relative to us. No! But what is the correct answer?

Consider the motion of the projectile in the lab frame: In the rocket frame, the projectile travels a distance $\Delta x'$ in time $\Delta t'$, thus it has a velocity observed in that frame of $v' = \Delta x'/\Delta t'$. The whole rocket frame moves relative to the lab frame at v_{rel}.

What is the velocity of the projectile in the lab frame? We must transform both the distance and time of the projectile's flight from the rocket frame into the lab frame and then compute the lab frame projectile velocity $v = \Delta x/\Delta t$.

[see p. 105 in the text]

The Lorentz transformation equations from $\Delta x', \Delta t'$ to $\Delta x, \Delta t$ are

$$\Delta x = \gamma \Delta x' + v_{rel}\gamma \Delta t'$$

$$\Delta t = \gamma \Delta t' + v_{rel}\gamma \Delta x'$$

Note the symmetry in these equations.

Now divide to get a velocity,

$$\frac{\Delta x}{\Delta t} = \frac{\gamma \Delta x' + v_{rel}\gamma \Delta t'}{\gamma \Delta t' + v_{rel}\gamma \Delta x'}$$

We can greatly simplify this equation. Every term in both the numerator and denominator is multiplied by γ, so just cancel those out. Next, divide both top and bottom by $\Delta t'$ to get

$$\frac{\Delta x}{\Delta t} = \frac{(\Delta x'/\Delta t') + v_{rel}}{1 + v_{rel}(\Delta x'/\Delta t')}$$

Notice that $\Delta x'/\Delta t'$ is just the projectile velocity in the rocket frame v'. Plug that in and the derivation yields simply

$$v = \frac{v' + v_{rel}}{1 + v' v_{rel}}$$

For our example $v = [0.6 + 0.5]/[1 + (0.6)(0.5)] = 0.85$ Not 1.1!

7 Causality: Why nothing travels faster than the speed of light and the Tale of the Tachyon Terror

[see Box L-1] 23rd Century America's Cup

Because of the rate at which technology changes, it is often postulated that one should wait until technology improves before beginning a project. For example, if we have a computer on which a job takes 1 year to run, but think that a new computer will become available in six months, on which the job will only take 3 months to run, then we should sit around and do nothing until we can get our hands on the faster computer. Imagine that a similar situation arises in a running of the America's Cup in the 23rd Century. Once a battle between sailing ships, it is a race between starships.

The destination is Alpha Centauri, 4 light years from Earth. The contestants in the final are the American defenders, flying the Enterprise, and the Russian challengers, in their yet-to-be-unveiled spacecraft, the Tachyon Terror.

The race begins and the Enterprise quickly accelerates up to 0.6c, speeding off toward Alpha Cen. But the Russian ship doesn't move. What are they thinking? Their ship isn't even ready! Their secret plan is to continue construction on their ship. Their scientists tell them that only a few years are needed to develop a faster than light propulsion mechanism. "Let the Americans waste their time! We'll blast right by them in a few years!"

After 4 years of construction, the Russian ship is ready. It launches at $v = 3c$. Wow! They pass the check points at 1 and 2 light years from Earth, then they catch up to the Enterprise in only 1 year, near the 3 light-year checkpoint, race by and win. Is this possible? What does this look like from the point of view of the Enterprise?

Let's draw a couple of spacetime diagrams to illustrate this. To make things easy, set the zero of the coordinate systems at the moment when the Tachyon Terror launches from Earth.

Introduction to spacetime diagrams: Here time is in years, distance in light years. Light travels at 45 degree angle, where $v = x/t = 1$.

First draw the spacetime diagram in the Earth frame, as planned by Russians. Events are

- TT leaves Earth $x = 0, t = 0$
- TT passes checkpoint 1 $x = 1, t = 1/3$
- TT passes checkpoint 2 $x = 2, t = 2/3$
- TT overtakes Enterprise $x = 3, t = 1$
- TT arrives at α Cen at $x = 4, t = 4/3$

Now draw in the Enterprise frame: We transform lab coordinates, as above, to rocket coordinates, as seen by the Enterprise. Where and when does the TT overtake the Enterprise? Apply the Lorentz transformation! Remember, the Enterprise moved at positive $0.6c$, thus the relative velocity of Earth frame as seen from Enterprise frame is negative. $v = 0.6c$, thus $\gamma = 1.25$

Launch of TT $x = 0, t = 0$, thus also $x' = 0, t' = 0$

Start of race is at $x = 0, t = -4$, thus

$$x' = \gamma x - v_{rel}\gamma t = (1.25)(0) - (0.6)(1.25)(-4) = 3$$

$$t' = -v_{rel}\gamma x + \gamma t = -(0.6)(1.25)(0) + (1.25)(-4) = -5$$

Note the time dilation effect!

Checkpoint 1 at $x = 1, t = 1/3$

$$x' = 1$$

$$t' = -1/3$$

Before launched!

Checkpoint 2 $x = 2, t = 2/3$

$$x' = 2$$

$$t' = -2/3$$

Before TT reaches checkpoint 1!

TT overtakes Enterprise $x = 3, t = 1$

$$x' = 3$$

$$t' = -1$$

Before the TT was built and launched.

TT reaches Alpha Cen $x = 4, t = 4/3$

$$x' = 4$$

$$t' = -4/3$$

From the Enterprise frame, the TT flies backwards from Alpha Cen, passes checkpoint 2 then 1 and lands at Earth just before its propulsion device was invented. Impossible! If the device already existed, why did they need to invent it?