Physics 326: Quantum Mechanics I
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MIDTERM EXAM Solutions

CLOSED-Book Problems
(total 30 points)

Problem C1: Properties of Stationary States (10 pts)

To get full credit, you must answer the following questions using the appropriate equation or equations (a sentence or so may be necessary, but do not fill the page with narrative).

(a) You find the stationary states by solving what differential equation? (Give its name exactly.) Now write down that differential equation, for the case of a particle moving in one dimension under the influence of a potential $V(x)$. (1 point + 3 points)

The time *independent* Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

(b) What is the uncertainty of the energy for a particle in a stationary state? (3 points)

Zero, because $\sigma_E = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$, but $H\psi_n = E_n\psi_n$, thus $\int \psi_n^* H^2 \psi_n dx = E^2$ and $\langle H\rangle = E$, thus $\sigma_E = 0$.

(c) If a particle is in a state that is a linear combination of properly-normalized stationary states, $\psi_n(x)$, at time $t = 0$,

$$\Psi(x, t = 0) = c_1\psi_1(x) + c_7\psi_7(x)$$

what equation must the coefficients c_1 and c_7 obey so that the wave function remains properly normalized? (3 points)

We need

$$1 = \int |\Psi(x,0)|^2 dx = \int [|c_1|^2|\psi_1(x)|^2 + |c_7|^2|\psi_7(x)|^2 + c_1^*\psi_1^*(x)c_7\psi_7(x) + c_7^*\psi_7^*(x)c_1\psi_1(x)]dx$$

By orthonormality of the ψ_n, the cross terms vanish and so we require

$$|c_1|^2 + |c_7|^2 = 1$$

Problem C2: What's Expected? (10 pts)

A particle moves in one dimension with wave function $\Psi(x,t)$. This wave function is not necessarily a stationary state.

(a) Write down (but do not solve), in all possible detail, an integral equation for the expectation value of its position x. Remember that the wave function may be complex. (4 points)

$$\langle x\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) dx = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx$$

(b) Write down (but do not solve), in all possible detail, an integral equation for the expectation value of its kinetic energy T. (4 points)

$$\langle T\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{T}\Psi(x,t)dx = \int_{-\infty}^{\infty} \Psi^*(x,t)\left(\frac{\hat{p}^2}{2m}\right)\Psi(x,t)dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{d^2}{dx^2}\Psi(x,t)dx$$

(c) Assuming that you have solved (a), write down an equation for the expectation value of the momentum, $\langle p\rangle$. (2 points)

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt}$$

Problem C3: Tests of your drawing skills (10 pts)

Carefully sketch the wavefunctions in parts (a) and (b), label the axes, and label each of the ψ_n in your plots with the correct quantum number n. Make certain that your plots of the wave functions have the correct behavior at $x = 0, \pm\infty$ and the correct number of nodes (zero crossings). Make two separate diagrams and make them large and neat.

(a) Draw a diagram of the wave functions $\psi_n(x)$ for the lowest three energy states of a particle in the one-dimensional potential well (potential $V(x) = \infty$ for $x < 0, x > a$). (5 points)

(b) Draw a diagram of the wave functions $\psi_n(x)$ for the lowest three energy states of a particle in the one-dimensional harmonic oscillator. (5 points)

See plots in Griffiths or any standard QM textbook.

OPEN-Book Problems

(total 30 points)

Problem O1: (10pts)

A particle in an infinite square well potential is in an initial state such that it has equal probability of being found anywhere in the well ($0 < x < a$).

(a) What is the properly-normalized wave function $\Psi(x, 0)$? (1 point)

$$\Psi(x,0) = \begin{cases} \sqrt{1/a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

This is obvious, but check using

$$1 = \int_0^a |\Psi(x,)|^2 dx = \frac{1}{a}\int_0^a dx = \frac{1}{a}a = 1$$

(b) Find the full time-dependent wave function $\Psi(x, t)$. (9 points)

$$\Psi(x,0) = \sum_n c_n \psi_n(x) = \sum_n c_n \sqrt{\frac{2}{a}} \sin(n\pi x/a)$$

where

$$c_n = \int_{-\infty}^{+\infty} \psi_n^*(x)\Psi(x,0)dx = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)\sqrt{\frac{1}{a}}dx = \frac{\sqrt{2}}{n\pi}\left[-\cos(n\pi x/a)\right]\Big|_0^a = \frac{2\sqrt{2}}{n\pi}\ n \,\text{odd only}$$

Thus, the complete wave function is

$$\Psi(x,t) = \frac{4}{\pi\sqrt{a}} \sum_{n=1,3,5\ldots} \frac{1}{n} \sin\left(\frac{n\pi x}{a}\right) e^{-iE_n t/\hbar}$$

with $E_n = n^2\pi^2\hbar^2/2ma^2$.

Problem O2: (10 pts)

A particle moves in a harmonic oscillator potential $V(x) = kx^2/2 = m\omega^2x^2/2$. At $t = 0$, the wave function of the particle is an even mix of the $n = 1$ and $n = 3$ stationary states,

$$\Psi(x,0) = A[\psi_1(x) + \psi_3(x)]$$

(a) What is the normalization constant A? (1 point)

We require

$$1 = \int |\Psi(x,0)|^2 dx = |A|^2 \int [\psi_1(x)+\psi_3(x)]^2 dx = \int [|\psi_1(x)|^2+|\psi_3(x)|^2+\psi_1^*(x)\psi_3(x)+\psi_3^*(x)\psi_1(x)]dx$$

By orthonormality of the ψ_n, $1 = 2|A|^2$, thus $A = 1/\sqrt{2}$.

(b) Compute the expectation value of the potential energy $V(x)$ at time $t = 0$ using what you learned about the raising and lowering operators in problem 2.12 (see solutions). (8 points)

Use $V(x) = m\omega^2x^2/2$ and we can write x in terms of ladder operators

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$$

We can write the x^2 part of the potential as

$$x^2 = \left(\frac{\hbar}{2m\omega}\right)(a_+^2 + a_+a_- + a_-a_+ + a_-^2)$$

and so

$$\begin{aligned}
\langle V \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,0)\hat{V}\Psi(x,0)dx \\
&= A^2\frac{m\omega^2}{2}\left(\frac{\hbar}{2m\omega}\right)\int_{-\infty}^{\infty}[\psi_1^*(x) + \psi_3^*(x)](a_+^2 + a_+a_- + a_-a_+ + a_-^2)[\psi_1(x) + \psi_3(x)]dx
\end{aligned}$$

Note carefully what happens to the various terms in the integrand: Terms with integrands of the form $\psi_n^* a_+^2 \psi_n$, or $\psi_n^* a_-^2 \psi_n$ die by orthonormality. The up/down terms

$$\int \psi_n^* a_+ a_- \psi_n dx = (n+1)$$

and down/up

$$\int \psi_n^* a_- a_+ \psi_n dx = n$$

survive, as do the terms that match ups or downs with the difference in n,

$$\int \psi_{n+2}^* a_+^2 \psi_n dx = \sqrt{(n+1)(n+2)}$$

(with $n = 1$)and

$$\int \psi_{n-2}^* a_-^2 \psi_n dx = \sqrt{n(n-1)}$$

(with $n = 3$). Plugging in to the integral, in that order,

$$\begin{aligned}\langle V \rangle &= \frac{\hbar\omega}{8}[1 + 3 + (1+1) + (3+1) + \sqrt{(1+1)(1+2)} + \sqrt{3(3-1)}] \\ &= \hbar\omega\left(\frac{5+\sqrt{6}}{4}\right)\end{aligned}$$

Note carefully that this is not the same as the average of the expectation values $\langle V \rangle$ of the two stationary states. That is because this is a mixed state, and so the expectation values may be a function of time! (cf. problem C1) In this particular case, we are asked to compute $\langle V \rangle$ at $t = 0$. Note what happens if we compute the expectation value of the potential as a function of time. Then the cross terms take the form

$$\int \psi_1^*(x) e^{+iE_1 t/\hbar} (a_-)^2 \psi_3(x) e^{-iE_3 t/\hbar} dx = \sqrt{6} e^{i(E_1 - E_3)t/\hbar}$$

and

$$\int \psi_3^*(x) e^{+iE_3 t/\hbar} (a_+)^2 \psi_1(x) e^{-iE_1 t/\hbar} dx = \sqrt{6} e^{i(E_3 - E_1)t/\hbar}$$

Using $E_3 - E_1 = \hbar\omega(7/2 - 3/2) = 2\hbar\omega$ and Euler's formula, the cross terms combine to contribute an oscillating term $(\hbar\omega/8)(2\sqrt{6}\cos(2\omega t))$. At $t = 0$, we get the result above. The time average of $\cos(2\omega t) = 0$, so the time average of $\langle V \rangle = (5/4)\hbar\omega$, but not at $t = 0$ (or, in general, at any specific time). Thus, Ehrenfest's theorem holds true but only in a time-averaged sense.

(c) At time $t > 0$, I make a measurement of the energy of the particle. What is the probability that I measure $E = (7/2)\hbar\omega$? (1 point)

That is the energy of the $n = 3$ state. The probabilities of measuring energy E_n is the square of the coefficient $|c_n|^2$, thus $P(E_3) = |c_3|^2 = 1/2$.

Problem O3: Harmonic Oscillator States (10 pts)

I can't for the life of me remember all the very important harmonic oscillator stationary states! Help me out by describing and providing formulae for two different ways of constructing them. Assume that I do remember that the ground state is $\psi_0 = \alpha \exp(-(m\omega/2\hbar)x^2)$. You can leave things in terms of $\alpha = (m\omega/\pi\hbar)^{1/4}$ and $\xi = (m\omega/\hbar)^{1/2}x$. If your method involves tricks for constructing the the Hermite polynomials, make sure that you also get the rest of the wave function correct. (5 points per method. Extra credit of 5 points for a third method, but it has to be completely right.)

First method: use the raising operator to act on the ground state:

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(a_+)^n \psi_0(x)$$

where

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x - \hbar\frac{d}{dx}\right)$$

Second method, use the recursion formula for the coefficients in the power series:

$$\psi_n(x) = h_n(\xi)e^{-\xi^2/2}$$

where

$$h_n(\xi) = \sum_j a_j \xi^j$$

and we use the recursion formula for the coefficients a_j,

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)}a_j$$

and remember that odd n have only odd terms ($a_0 = 0$) and even n have only even terms ($a_1 = 0$).

Third method: If you remember that

$$\psi_n(x) = \alpha\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$$

then you can apply the Rodrigues formula

$$H_n(\xi) = (-1)^n\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

to get the Hermite polynomials. If you didn't remember the coefficients in front, you could always get them by normalizing each wavefunction yourself by integration.

Fourth method: Same assumptions as the third, but now using the generating function approach to get the Hermite polynomials: $H_n(\xi)$ is the nth derivative with respect to z of the generating function $\exp(-z^2 + 2z\xi)$.

Fifth method: If you also remember $\psi_1(x)$ or that $H_1(\xi) = 2\xi$ (we'll take it for granted that you remember $H_0 = 1$), then use same assumptions as the third and fourth methods, but use the following recursion relation for the H_n:

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$$

Bonus Problem Quantum or Classical? (5 pts)

A mass m slides on a frictionless horizontal plane and is attached to a massless spring with spring constant k. The mass is pulled a distance x_0 from the equilibrium position and released. Compare the total energy of this system to the energy difference between consecutive energy levels to compute the energy quantum number of the system, $n = E/\Delta E$. Use this to derive a condition on m, k and x_0 for this system to behave classically.

The total energy is $kx_0^2/2$ and the gap between energy levels is $\hbar\omega$, where $\omega = \sqrt{k/m}$, thus n is of order

$$n = \frac{E}{\Delta E} = \frac{kx_0^2/2}{\hbar\omega} = \frac{kx_0^2/2}{\hbar\omega} = \frac{\sqrt{km}x_0^2}{2\hbar}$$

The system will behave classically if $n \gg 1$. Unsurprisingly, this occurs for large mass, large spring constant, or large initial displacement x_0.