Physics 326: Quantum Mechanics I
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Problem Set 5 Solutions

Problem 1

Griffiths 2.18

Show that $Ae^{ikx} + Be^{-kx}$ and $C\cos kx + D\sin kx$ are equivalent ways of writing the same function of x and determine the constants C and D in terms of A and B, and vice versa.

Use Euler's formula to write

$$\begin{aligned} Ae^{ikx} + Be^{-kx} &= A(\cos kx + i\sin kx) + B(\cos kx - i\sin kx) \\ &= (A+B)\cos kx + i(A-B)\sin kx \\ &= C\cos kx + D\sin kx \end{aligned}$$

with $C = A + B$ and $D = i(A - B)$. Likewise,

$$\begin{aligned} C\cos kx + D\sin kx &= C\left(\frac{e^{ikx}+e^{-ikx}}{2}\right) + D\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right) \\ &= \frac{1}{2}(C-iD)e^{ikx} + \frac{1}{2}(C+iD)e^{-ikx} \\ &= Ae^{ikx} + Be^{-kx} \end{aligned}$$

with $A = (C - iD)/2$ and $B = (C + iD)/2$.

Problem 2

Griffiths 2.19

Find the probability current, J, for the free particle wave function equation 2.94,

$$\Psi_k(x,t) = Ae^{i(kx-\frac{\hbar k^2}{2m}t)}$$

In which direction does the probability current flow?

$$\begin{aligned} J &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\ &= \frac{i\hbar}{2m}|A|^2\left[e^{i(kx-\hbar k^2t/2m)}(-ik)e^{-i(kx-\hbar k^2t/2m)} - e^{-i(kx-\hbar k^2t/2m)}(ik)e^{i(kx-\hbar k^2t/2m)}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{i\hbar}{2m}|A|^2(-2ik) \\
&= \frac{\hbar k}{m}|A|^2
\end{aligned}
$$

This flows in the positive x direction, as expected.

Problem 3

Griffiths 2.20

Steps to a "proof" of Plancherel's theorem.

(a) Dirichlet's theorem says that any function $f(x)$ on the interval $[-a, +a]$ can be expanded as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty}[a_n \sin(n\pi x/a) + b_n \cos(n\pi x/a)]$$

Show that that is equivalent to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$$

What is c_n in terms of a_n and b_n?

First, note that the $n = 0$ term in the sum is just b_0. Apply Euler's formula yet again,

$$
\begin{aligned}
f(x) &= b_0 + \sum_{n=1}^{\infty}\frac{a_n}{2i}\left(e^{in\pi x/a} - e^{-in\pi x/a}\right) + \sum_{n=1}^{\infty}\frac{b_n}{2}\left(e^{in\pi x/a} + e^{-in\pi x/a}\right) \\
&= b_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{in\pi x/a} + \sum_{n=1}^{\infty}\left(-\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{-in\pi x/a}
\end{aligned}
$$

If we let $c_0 = b + 0$ and $c_n = (-ia_n + b_n)/2$ for $n = 1, 2, 3...$ and $c_n = (ia_{-n} + b_{-n})/2$ for $n = -1, -2, -3...$, then

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$$

as required.

(b) Modify Fourier's trick to show that

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx$$

$$\int_{-a}^{a} f(x)e^{-im\pi x/a}dx = \sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{i(n-m)\pi x/a}dx$$

For $n = m$, this is obviously just $2ac_m$, but for $n \neq m$,

$$\int_{-a}^{a} e^{i(n-m)\pi x/a} dx = \left. \frac{e^{i(n-m)\pi x/a}}{i(n-m)\pi/a} \right|_{-a}^{a} = 0$$

Thus, all terms with $n \neq m$ are zero and

$$\int_{-a}^{a} f(x) e^{-im\pi x/a} dx = 2ac_m$$

and so

$$c_n = \frac{1}{2a} \int_{-a}^{a} f(x) e^{-in\pi x/a} dx$$

(c) Eliminate n and c_n in favor of new variables $k = n\pi/a$ and $F(k) = \sqrt{2/\pi} a c_n$. Show that (a) and (b) become

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k) e^{ikx} \Delta k$$

where

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}$$

and Δk is the increment in k from one n to the next.

Write the Fourier expansion of $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{a} F(k) e^{ikx} = \frac{1}{\sqrt{2\pi}} \sum F(k) e^{ikx} \Delta k$$

where $\Delta k = \pi/a$, and the function $F(k)$ as

$$F(k) = \sqrt{\frac{2}{\pi}} a \frac{1}{a} \int_{-a}^{a} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ikx} dx$$

(d) Now take the limit $a \to \infty$ to obtain Plancherel's Theorem.

As $a \to \infty$, k becomes a continuous variable and so the sum in (c) becomes an integral,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

with the "coefficients" being the Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Problem 4

Griffiths 2.22

A free particle has initial wave function

$$\Psi(x,0) = Ae^{-ax^2}$$

where A and a are constants (and a is both real and positive). This is a "gaussian wave packet."

(a) Normalize $\Psi(x,0)$.

$$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = |A|^2 \sqrt{\frac{\pi}{2a}}$$

Thus, $A = (2a/\pi)^{1/4}$.

(b) Find $\Psi(x,t)$. See hint in Griffiths about "completing the square" in the exponent.

To find the full time evolution,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \hbar k^2 t/2m)} dk$$

we need to compute

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{i(kx-\hbar k^2 t/2m)} = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{i(kx-\hbar k^2 t/2m)} dx$$

To evaluate the integral, follow the suggestion: Define $y = \sqrt{a}[x + (b/2a)]$, thus $(ax^2 + bx) = y^2 - (b^2/4a)$. Using this,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-ax^2+bx} dx &= \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)} \frac{1}{\sqrt{a}} dy \\
&= \frac{1}{sqrta} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy \\
&= \sqrt{\frac{\pi}{a}} e^{b^2/4a}
\end{aligned}$$

Now we see that

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{i(kx-\hbar k^2 t/2m)} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} e^{-k^2/4a} \\
&= \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}
\end{aligned}$$

where we just substituted $b = ik$ above.

Now for the really fun part. We can pull the same trick when we write the full, time-evolving wave function,

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\hbar k^2t/2m)}dk \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty}e^{-k^2/4a}e^{i(kx-\hbar k^2t/2m)}dk\end{aligned}$$

Combine the exponents and write that as $(1/4a + i\hbar t)k^2 - ixk$ to get it in the form above so you can use the integral solution. The coefficients are a mess, but they simplify,

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\frac{\sqrt{\pi}}{\sqrt{(1/4a)+(i\hbar t/2m)}}e^{-x^2/4((1/4a)+(i\hbar t/2m))} \\ &= \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+2i\hbar at/m}}e^{-ax^2/(1+2i\hbar at/m)}\end{aligned}$$

Wow. A closed form solution for the time evolution! Note that it is extremely rare to be able to do this.

(c) Find $|\Psi(x,t)|^2$ and express your answer in terms of

$$w \equiv \sqrt{\frac{a}{1+(2\hbar at/m)^2}}$$

Now sketch $\Psi|^2$ as a function of x at $t = 0$ and, again, for t very large. Qualitively, what happens as time goes on?

Note carefully that to form $|\Psi|^2$ you must complex conjugate the stuff in the square root. To speed things up, define $\theta = 2\hbar at/m$. Thus

$$\begin{aligned}|\Psi|^2 &= \sqrt{\frac{2a}{\pi}}\frac{e^{-ax^2/(1+i\theta)}e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}} \\ &= \sqrt{\frac{2a}{\pi}}\frac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}\end{aligned}$$

Using the definition of w suggested above,

$$|\Psi|^2 = \sqrt{\frac{2}{\pi}}we^{-2w^2x^2}$$

As t increases, $|\Psi|^2$ spreads out and flattens; the wave packet becomes more dispersed with time.

(d) Find the expectation values $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, and the uncertainties σ_x and σ_p.

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = 0$$

because the integrand is odd.

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = 0.$$

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}}w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx = \sqrt{\frac{2}{\pi}}w\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}} = \frac{1}{4w^2}$$

Note that w depends on time, so the expectation value $\langle x^2 \rangle$ depends on time; again, the wave packet becomes broader with time.

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{d^2\Psi}{dx^2} dx$$

Note that we are interested in the time evolution, so you can't set $t = 0$ and then solve the much simpler problem using $\Psi(x,)$. Believe it or not, the result of the integral simplifies to

$$\langle p^2 \rangle = a\hbar^2$$

$$\sigma_x = \frac{1}{2w}$$

$$\sigma_p = \sqrt{a}\hbar$$

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

$$\sigma_x\sigma_p = \frac{1}{2w}\hbar\sqrt{a} = \frac{\hbar}{2}\sqrt{1 + (2\hbar at/m)^2} \geq \frac{\hbar}{2}$$

which has minimum uncertainty at $t = 0$ when it is exactly at the limit.