Physics 326: Quantum Mechanics I
Prof. Michael S. Vogeley
Fall 2009

Problem Set 4 Solutions

Problem 1

Griffiths 1.17

A particle has initial wave function, at time $t = 0$,

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2), & -a \le x \le +a \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the normalization constant A.

$$\begin{aligned} 1 &= |A|^2 \int_{-a}^{a} (a^2 - x^2) dx \\ &= 2|A|^2 \int_{0}^{a} (a^4 - 2a^2x^2 + x^4) dx \\ &= 2|A|^2 (a^4 x - 2a^2x^3/3 + x^5/5)|_0^a \\ &= \frac{16}{15} a^5 |A|^2 \end{aligned}$$

Thus, $A = \sqrt{15/16a^5}$.

(b) What is the expectation value of x at $t = 0$?

The integrand of

$$\langle x \rangle = \int_{-a}^{a} x |\Psi|^2 dx$$

is odd, thus $\langle x \rangle = 0$.

(c) What is the expectation valule of p at $t = 0$? (Note that you cannot compute this from $p = md\langle x \rangle/dt$. Why not?)

You need to compute the expectation value of the operator $\hat{p} = (\hbar/i)d/dx$,

$$\langle p \rangle = \frac{\hbar}{i} A^2 \int_{-a}^{a} (a^2 - x^2) \frac{d}{dx} (a^2 - x^2) dx = \frac{\hbar}{i} A^2 \int_{-a}^{a} (a^2 - x^2)(-2x) dx = 0$$

because, again, the integrand is odd.

You can't use the "easy" way, because you only know $\langle x \rangle$ at $t = 0$ and so you can't compute $d\langle x \rangle/dt$.

(d) Find $\langle x^2 \rangle$.

$$\begin{aligned} \langle x^2 \rangle &= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 dx = 2A^2 \int_0^a (a^4 x^2 - 2a^2 x^4 + x^6) dx \\ &= 2\frac{15}{16a^5} \left[a^4 \frac{x^3}{3} - 2a^2 \frac{x^5}{5} + \frac{x^7}{7} \right] \Bigg|_0^a = \frac{a^2}{7} \end{aligned}$$

(e) Find $\langle p^2 \rangle$.

$$\begin{aligned} \langle p^2 \rangle &= -A^2 \hbar^2 \int_{-a}^{a} (a^2 - x^2) \frac{d^2}{dx^2} (a^2 - x^2) dx = 2A^2 \hbar^2 2 \int_0^a (a^2 - x^2) dx \\ &= 4\frac{15}{16a^5} \hbar^2 \left(a^2 x - \frac{x^3}{3} \right) \Bigg|_0^a = \frac{5}{2} \frac{\hbar^2}{a^2} \end{aligned}$$

(f) Find the uncertainty in x (σ_x).

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{\sqrt{7}}$$

(g) Find the uncertainty in p (σ_p).

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5}{2}} \frac{\hbar}{a}$$

(h) Check that your results in (f) and (g) are consistent with the uncertainty principle.

$$\sigma_x \sigma_p = \frac{a}{\sqrt{7}} \sqrt{\frac{5}{2}} \frac{\hbar}{a} = \sqrt{\frac{10}{7}} \frac{\hbar}{2} > \frac{\hbar}{2}$$

Problem 2

Griffiths 2.8

A particle of mass m in the infinite square well of width a starts out in the left half of the well, with equal probability of being found at any point in that half.

(a) What is the initial wave function $\Psi(x, 0)$? (Assume it is real and don't forget to normalize it.)

$$\Psi(x,0) = \begin{cases} A, & 0 < x < a/2 \\ 0, & otherwise. \end{cases}$$

with normalization

$$1 = A^2 \int_0^{a/2} dx = A^2 \frac{a}{2}$$

thus $A = \sqrt{2/a}$.

(b) What is the probability that a measurement of the energy would yield the value $\pi^2\hbar^2/2ma^2$?

This is the energy of the $n = 1$ state, $\psi_1(x) = \sqrt{2/a}\sin(\pi x/a)$. The probability of measuring this energy is $|c_1|^2$, where

$$c_1 = A\sqrt{\frac{2}{a}} \int_0^{a/2} \sin(\pi x/a)dx = \frac{2}{a}\left[-\frac{a}{\pi}\cos(\pi x/a)\right]\Big|_0^{a/2} = \frac{2}{\pi}$$

Thus, $P_1 = (2/\pi)^2 = 0.4053$.

Problem 3

Griffiths 2.14

A particle is in the ground state of the harmonic oscillator with classical frequency ω. Suddenly, the spring constant quadruples, so that the frequency doubles, $\omega' = 2\omega$ without disturbing the initial wave function. (Of course, now Ψ will evolve differently, because the Hamiltonian has changed.) What is the probability that a measurement of the energy would still return the value $\hbar\omega/2$? What is the probability of measuring energy $\hbar\omega$?

The energies of the new oscillator potential are $E_n = \hbar\omega'(n + 1/2) = 2\hbar\omega(n + 1/2)$. Thus, the new ground state, $n = 0$, has energy $\hbar\omega$, so the probability of measuring half that energy is identically zero.

The initial wave function is the old ground state,

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}}x^2$$

The new ground state is

$$\psi_0'(x) = \left(\frac{2m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{2m\omega}{2\hbar}}x^2$$

The probability of measuring the new ground state energy given the initial state above is $P_0 = |c_0|^2$, where

$$\begin{aligned} c_0 &= \int \psi_0'(x)\psi_0(x) \\ &= 2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{3m\omega}{2\hbar}}x^2 dx \\ &= 2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}} 2\sqrt{\pi}\left(\frac{1}{2}\sqrt{\frac{2\hbar}{3m\omega}}\right) \\ &= 2^{1/4}\sqrt{\frac{2}{3}} \end{aligned}$$

thus $P_0 = 2\sqrt{2}/3 = 0.9428$.

Problem 4

Griffiths 2.17

Useful theorems involving Hermite polynomial

(a) Use the Rodrigues formula,

$$H_n(\xi) = (-1)^n \left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

to derive H_3 and H_4.

$$H_3(\xi) = -e^{\xi^2}\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = -12\xi + 8\xi^3$$

$$H_4(\xi) = e^{\xi^2}\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = 12 - 48\xi^2 + 16\xi^4$$

(b) Use the recursion relation,

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$$

and your answer to (a) to obtain H_5 and H_6.

$$H_5 = 2\xi H_4 - 8H_3 = 2\xi(12\xi - 48\xi^3 + 16\xi^4) - 8(-12\xi + 8\xi^3) = 120\xi - 160\xi^3 + 32\xi^5$$

$$H_6 = 2\xi H_5 - 10H_4 = 2\xi(120\xi - 160\xi^3 + 32\xi^5) - 10(12\xi - 48\xi^3 + 16\xi^4) = -120 + 720\xi^2 - 480\xi^4 + 64\xi^6$$

(c) If you differentiate a nth order polynomial, you always get a polynomical of order $(n-1)$, for example,

$$\frac{dH_n(\xi)}{d\xi} = 2nH_{n-1}(\xi)$$

Check this, by differentiating H_5 and H_6.

$$\frac{dH_5}{d\xi} = 120 - 480\xi^2 + 160\xi^4 = (2)(5)H_4$$

$$\frac{dH_6}{d\xi} = 1440\xi - 1920\xi^3 + 384\xi^5 = (2)(6)H_5$$

(d) $H_n(\xi)$ is the nth derivative with respect to z of the generating function $exp(-z^2+2z\xi)$, with $z = 0$. Use this to derive H_0, H_1 and H_2.

Taking the zeroth derivative (don't differentiate at all) and setting $z = 0$, $H_0 = 1$.

$$\frac{d}{dz}e^{-z^2+2z\xi} = (-2z + \xi)e^{-z^2+2z\xi}$$

Setting $z = 0$ yields $H_1(\xi) = 2\xi$

Differentiate again,

$$\frac{d^2}{dz^2}e^{-z^2+2z\xi} = [-2 + (-2z + 2\xi)^2]e^{-z^2+2z\xi}$$

Setting $z = 0$ yields $H_2(\xi) = -2 + 4\xi^2$