

**Physics 326: Quantum Mechanics I**  
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**Fall 2009**

**Problem Set 3 Solutions**

**Problem 1**

Griffiths 2.11

(a) Compute  $\langle x \rangle$ ,  $\langle p \rangle$ ,  $\langle x^2 \rangle$  and  $\langle p^2 \rangle$  for the states  $\psi_0$  and  $\psi_1$  by explicit integration (in other words, don't do problem 2.12 first and then plug in values of  $n$ ). Hint: it simplifies things to define  $\xi \equiv \sqrt{m\omega\hbar}x$  and  $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$ .

Using the suggested definitions,

$$\psi_0(x) = \alpha e^{-\xi^2/2}, \quad \psi_1(x) = \sqrt{2}\alpha\xi e^{-\xi^2/2}$$

To compute  $\langle x \rangle$ , note that  $|\psi|^2$  is an even function, thus  $\int x|\psi|^2 dx$  has an odd integrand and  $\langle x \rangle = 0$  for both  $n = 0, 1$ .

Using  $\langle p \rangle = m d\langle x \rangle / dt$ , we immediately find  $\langle p \rangle = 0$  for both states.

For the  $n = 0$  state,

$$\begin{aligned} \langle x^2 \rangle &= \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\xi^2} dx \\ &= \alpha^2 \left( \frac{\hbar}{m\omega} \right)^{3/2} \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi \\ &= \frac{1}{\sqrt{\pi}} \frac{\hbar}{m\omega} \frac{\sqrt{\pi}}{2} \\ &= \frac{\hbar}{2m\omega} \end{aligned}$$

$$\begin{aligned} \langle p^2 \rangle &= \psi_0^* \left( \frac{\hbar}{i} \frac{d}{dx} \right)^2 \psi_0 dx \\ &= -\hbar^2 \alpha^2 \sqrt{\frac{m\omega}{\hbar}} \int_{-\infty}^{\infty} e^{-\xi^2/2} \frac{d^2}{d\xi^2} e^{-\xi^2/2} d\xi \\ &= -\frac{m\hbar\omega}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\xi^2 - 1) e^{-\xi^2} d\xi \\ &= -\frac{m\hbar\omega}{\sqrt{\pi}} (\sqrt{\pi}/2 - \sqrt{\pi}) \end{aligned}$$

$$= \frac{m\hbar\omega}{2}$$

For the  $n = 1$  state,

$$\begin{aligned}\langle x^2 \rangle &= 2\alpha^2 \int_{-\infty}^{\infty} x^2 \xi^2 e^{-\xi^2} dx \\ &= 2\alpha^2 \left( \frac{\hbar}{m\omega} \right)^{3/2} \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi \\ &= \frac{2\hbar}{\sqrt{\pi}m\omega} \frac{3\sqrt{\pi}}{4} \\ &= \frac{3\hbar}{2m\omega}\end{aligned}$$

$$\begin{aligned}\langle p^2 \rangle &= \int \psi_1^* \left( \frac{\hbar}{i} \frac{d}{dx} \right)^2 \psi_1 dx \\ &= -2\hbar^2 \alpha^2 \sqrt{\frac{m\omega}{\hbar}} \int_{-\infty}^{\infty} \xi e^{-\xi^2/2} \frac{d^2}{d\xi^2} (\xi e^{-\xi^2/2}) d\xi \\ &= -\frac{2m\hbar\omega}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\xi^4 - 3\xi^2) e^{-\xi^2} d\xi \\ &= -\frac{2m\hbar\omega}{\sqrt{\pi}} \left( \frac{3}{4}\sqrt{\pi} - \frac{3}{2}\sqrt{\pi} \right) \\ &= \frac{3m\hbar\omega}{2}\end{aligned}$$

(b) Check the uncertainty principle for these states.

For  $n = 0$ ,

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \\ \sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\hbar\omega}{2}}\end{aligned}$$

Thus

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

which is exactly the minimum uncertainty.

For  $n = 1$ ,

$$\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}}, \quad \sigma_p = \sqrt{\frac{3m\hbar\omega}{2}}$$

Thus

$$\sigma_x \sigma_p = \frac{3\hbar}{2}$$

which is indeed larger than the minimum.

(c) Compute  $\langle T \rangle$ , the expectation value of kinetic energy, and  $\langle V \rangle$ , the expectation value of potential energy, for these states. Is their sum what you would expect?

$$\langle T \rangle = \frac{1}{2m} \langle p^2 \rangle = \begin{cases} \frac{1}{4} \hbar \omega, & n = 0 \\ \frac{3}{4} \hbar \omega, & n = 1 \end{cases}$$

$$\langle V \rangle = \frac{1}{2} m \omega^2 \langle x^2 \rangle = \begin{cases} \frac{1}{4} \hbar \omega, & n = 0 \\ \frac{3}{4} \hbar \omega, & n = 1 \end{cases}$$

Thus the sum of these energies is

$$\langle T \rangle + \langle V \rangle = \langle H \rangle = \begin{cases} \frac{1}{2} \hbar \omega = E_0, & n = 0 \\ \frac{3}{2} \hbar \omega = E_1, & n = 1 \end{cases}$$

as expected.

## Problem 2

Griffiths 2.12

Find  $\langle x \rangle$ ,  $\langle p \rangle$ ,  $\langle x^2 \rangle$ ,  $\langle p^2 \rangle$ , and  $\langle T \rangle$  for the  $n$ th stationary states of the harmonic oscillator, using the method of example 2.5. Check that the uncertainty principle is satisfied.

Using the ladder operators, we can write

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-), \quad p = i\sqrt{\frac{\hbar m\omega}{2}} (a_+ - a_-)$$

Thus,

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \int \psi_n^* (a_+ + a_-) \psi_n dx$$

Using

$$a_+ \psi_n = \sqrt{n+1} \psi_{n+1}, \quad a_- \psi_n = \sqrt{n} \psi_{n-1}$$

we obtain

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left[ \sqrt{n+1} \int \psi_n^* \psi_{n+1} dx + \sqrt{n} \int \psi_n^* \psi_{n-1} dx \right] = 0$$

by the orthogonality of the stationary states  $\psi_n$ .

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = 0$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \int \psi_n^* (a_+^2 + a_+ a_- + a_- a_+ + a_-^2) \psi_n dx$$

but following the results above, we see that

$$\begin{aligned} a_+^2 \psi_n &= \sqrt{(n+1)(n+2)} \psi_{n+2} \\ a_+ a_- \psi_n &= n \psi_n \\ a_- a_+ \psi_n &= (n+1) \psi_n \\ a_-^2 \psi_n &= \sqrt{(n-1)n} \psi_{n-2} \end{aligned}$$

Using these results and the orthonormality of the the  $\psi_n$ , we obtain

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} [0 + n + (n+1) + 0] = \frac{\hbar}{2m\omega} (2n+1) = \frac{\hbar}{m\omega} (n + 1/2)$$

We can follow the same approach to obtain the other expectation values:

$$\begin{aligned} p^2 &= -\frac{\hbar m\omega}{2} (a_+ - a_-)^2 = -\frac{\hbar m\omega}{2} (a_+^2 - a_+ a_- - a_- a_+ + a_-^2) \\ \langle p^2 \rangle &= -\frac{\hbar m\omega}{2} [0 - n - (n+1) + 0] = m\hbar\omega \left( n + \frac{1}{2} \right) \\ \langle T \rangle &= \frac{1}{2m} \langle p^2 \rangle \\ &= \frac{1}{2} m\hbar\omega \left( n + \frac{1}{2} \right) \end{aligned}$$

The uncertainties are

$$\sigma_x = \sqrt{n + \frac{1}{2}} \sqrt{\frac{\hbar}{m\omega}}, \quad \sigma_p = \sqrt{n + \frac{1}{2}} \sqrt{m\hbar\omega}$$

thus

$$\sigma_x \sigma_p = \left( n + \frac{1}{2} \right) \hbar \geq \frac{\hbar}{2}$$

### Problem 3

Griffiths 2.15

In the ground state of the harmonic oscillator, what is the probability (accurate to three significant digits) of finding the particle outside the classically allowed region? Hint: Classically, the energy of an oscillator is  $E = (1/2)ka^2 = (1/2)m\omega^2a^2$ , where  $a$  is the amplitude. So the “classically allowed region” extends from  $x = -\sqrt{2E/m\omega^2}$  to  $+\sqrt{2E/m\omega^2}$ . You can look up the numerical value of the integral in a table under “Normal Distribution” or “Error Function.”

The ground state is

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$$

This is symmetric about zero, so we can compute the probability of finding the particle beyond  $\pm x_0$  as

$$P = 2\sqrt{\frac{m\omega}{\pi\hbar}} \int_{x_0}^{\infty} e^{-\xi^2} dx = 2\sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\hbar}{m\omega}} \int_{\xi_0}^{\infty} e^{-\xi^2} d\xi$$

after we change variables from  $x$  to  $\xi$ . The classically allowed region extends out to where the potential energy is  $(1/2)m\omega x_0^2 = E_0 = \hbar\omega/2$ , thus  $\xi_0 = 1$ . Now look up

$$P = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-\xi^2} d\xi = 2(1 - F(\sqrt{2})) = 0.157$$

where  $F$  is the so-called error function.

### Problem 4

Griffiths 2.42

Find the allowed energies of the *half* harmonic oscillator, which has potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2, & x > 0 \\ \infty, & x < 0 \end{cases}$$

This represents, for example, a spring that can be stretched but not compressed. Think! This requires very little computation.

All of the work we did to find the stationary states of the harmonic oscillator apply here *plus* a new boundary condition:  $\psi(0) = 0$ . This condition eliminates all of the even solutions ( $n = 0, 2, 4, \dots$ ). The odd solutions are OK because they all go to zero at the origin, thus the allowed energies are simply

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right), n = 1, 3, 5, \dots$$