Physics 326: Quantum Mechanics I
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Problem Set 1 Solutions

Problem 1

Griffiths 1.5

Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A, λ and ω are real, positive constants.

(a) Normalize Ψ.

Use the fact that the wavefunction is even and look up the definite integral.

$$1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = 2A^2 \int_0^{\infty} e^{-2\lambda x} dx = 2A^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right) = \frac{A^2}{\lambda}$$

Thus, $A = \sqrt{\lambda}$.

(b) Compute the expectation values $\langle x \rangle$ and $\langle x^2 \rangle$.

$$\langle x \rangle = \int x|\Psi|^2 dx = A^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|}dx = 0$$

because the integrand is odd.

Again using the even nature of the wave function to "double over" the integrals,

$$\langle x^2 \rangle = 2A^2 \int_0^{\infty} x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \frac{1}{2\lambda^2}$$

(c) Compute the standard deviation of x, σ_x. Also sketch the graph of $|\Psi|^2$ and mark the location of points $(\langle x \rangle + \sigma)$ and $(\langle x \rangle - \sigma)$. Compute the probability that the particle would be found outside this range.

Using results from above,

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}$$

The values of $|\Psi|^2$ at $\langle x\rangle \pm \sigma_x$ are simply

$$|\Psi(\pm\sigma_x)|^2 = A^2 e^{-2\lambda\sigma_x} = \lambda e^{-\sqrt{2}} = 0.2431\lambda$$

The probability outside $x = \pm\sigma_x$ is the integral over the tails of the distribution, which you again can evaluate by using symmetry. Thus the probability is

$$P = 2\int_{\sigma_x}^{\infty} |\Psi|^2 dx = 2A^2 \int_{\sigma_x}^{\infty} e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Bigg|_{\sigma_x}^{\infty} = e^{-2\lambda\sigma_x} = e^{-\sqrt{2}} = 0.2431.$$

Problem 2

Griffiths 1.8

Suppose you add a constant potential V_0 to the potential energy. Show that the wave function picks up a time-dependent phase factor: $\exp(-iV_0 t/\hbar)$ (Hint: Do this by substitution of the modified wave function into the Schrödinger equation.). What effect does this have on the value of a dynamical variable?

If the wave function Ψ is a solution to

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

Replace Ψ with $\Psi_0 = \Psi\exp(-iV_0 t)$ and you (almost) immediately see that

$$i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V + V_0)\Psi_0$$

thus $\Psi_0 = \Psi\exp(-iV_0 t)$ is a solution for the case of an extra constant potential.

The extra time-dependent phase factor has no effect on the expectation values of dynamical variables.

Problem 3

Griffiths 1.11

The needle on a car speedometer is broken and is free to swing over the range of angle from 0 to π.

(a) What is the probability density $\rho(\theta)$? Make sure that you properly normalize $\rho(\theta)$ and remember that $\rho(\theta)$ is the probability that the needle lands in the infinitesimal range $(\theta, \theta + d\theta)$. Graph $\rho(\theta)$ over the range $(-\pi/2, 3\pi/2)$.

$$x = \begin{cases} 1/\pi & \text{if } 0 \leq \theta \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute the expectation values $\langle\theta\rangle, \langle\theta^2\rangle$, and the standard deviation, σ_θ.

$$\langle\theta\rangle = \int \theta\rho(\theta)d\theta = \frac{1}{\pi}\int_0^\pi \theta d\theta = \frac{1}{\pi}\left(\frac{\theta^2}{2}\right)^2\Bigg|_0^\pi = \frac{\pi}{2}$$

which is a lot of work to state the obvious.

$$\langle\theta^2\rangle = \int \theta^2\rho(\theta)d\theta = \frac{1}{\pi}\int_0^\pi \theta^2 d\theta = \frac{1}{\pi}\left(\frac{\theta^3}{3}\right)^2\Bigg|_0^\pi = \frac{\pi^2}{3}$$

$$\sigma_\theta^2 = \langle\theta^2\rangle - \langle\theta\rangle^2 = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi}{2\sqrt{3}}$$

(c) Compute the expectation values $\langle\sin\theta\rangle, \langle\cos\theta\rangle$, and $\langle\cos^2\rangle$.

$$\langle\sin\theta\rangle = \int \sin\theta\rho(\theta)d\theta = \frac{1}{\pi}\int_0^\pi \sin\theta d\theta = \frac{1}{\pi}(-\cos\theta)|_0^\pi = \frac{2}{\pi}$$

$$\langle\cos\theta\rangle = \int \cos\theta\rho(\theta)d\theta = \frac{1}{\pi}\int_0^\pi \cos\theta d\theta = \frac{1}{\pi}(\sin\theta)|_0^\pi = 0$$

$$\langle\cos^2\theta\rangle = \int \cos^2\theta\rho(\theta)d\theta = \frac{1}{\pi}\int_0^\pi \cos^2\theta d\theta = \frac{1}{\pi}\int_0^\pi (1/2)\theta d\theta = \frac{1}{2}$$

where we used the trick that $\sin^2\theta + \cos^2\theta = 1$ and $\int \sin^2\theta d\theta = \int \cos^2\theta d\theta$ over this range.

Problem 4

Griffiths 1.14

Let $P_{ab}(t)$ be the probability of finding a particle in the range $a < x < b$ at time t.

(a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$

is called the "probability current," which measures the flow of probability past the point x.

$$P_{ab} = \int_a^b |\Psi(x,t)|^2 dx$$

thus

$$\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi(x,t)|^2 dx$$

Now recall our result (see eq. 1.125 in Griffiths) that

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)\right] = -\frac{\partial}{\partial x}J(x,t)$$

using J as defined above. Now taking the integral is easy,

$$\frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t) = -J(x,t)|_a^b = J(a,t) - J(b,t)$$

What are the units of $J(x,t)$?

Probability is dimensionless, so J must have units of inverse seconds.

(b) Compute the probability current for the wave function

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]}$$

Factor out the time-dependent part, since you can immediately see that this will drop out when you multiply by the complex conjugate. Now just deal with $\psi(x)$, and you easily see that

$$\psi^*\frac{d\psi}{dx} = \frac{d\psi^*}{dx}\psi$$

because ψ is real, so $J(x,t) = 0$.

Problem 5

Griffiths 1.18

In general, quantum mechanics is relevant (and, therefore, necessary!) when the de Broglie wavelength of a particle, $\lambda = h/p$, is larger than the characteristic size of the physical system, d. Consider that in thermal equilibrium the average kinetic energy of a particle is

$$\frac{p^2}{2m} = \frac{3}{2} k_B T$$

where k_B is Boltzmann's constant. Using this, the de Broglie wavelength is of order

$$\lambda = \frac{h}{\sqrt{3mk_BT}}$$

Now, figure out whether or not the following cases must be treated quantum mechanically.

(a) Solids.

The lattice spacing in solid is typically of order $d = 0.3$ mm. Below what temperature are the free electrons quantum mechanical? Below what temperature are the nuclei quantum mechanical? (Use nuclei of sodium as a test case.) Hint: You should find that electrons are always quantum mechanical, but nuclei are almost never quantum mechanical.

The system is quantum mechanical if $\lambda > d$, or

$$\frac{h}{\sqrt{3mk_BT}} > d$$

thus the temperature must be below

$$T < \frac{h^2}{2mk_Bd^2}$$

Using $m_e = 9.31 \times 10^{-31}$ kg, $k_B = 1.4 \times 10^{-23}$J/K, and $h = 6.6 \times 10^{-34}$ J s, $T < 1.3 \times 10^5$ K for electrons.

For sodium nuclei, $m = 23m_p = 3.9 \times 10^{-26}$ kg, thus $T < 3.0K$.

So, under most circumstances you'll deal with, electrons are quantum mechanical and nuclei are not (we'll use this later when we study atoms and molecules!).

(b) Gases.

For what temperature are atoms in an ideal gas at pressure P quantum mechanical? Hint: Use $PV = Nk_BT$ to estimate the interatomic spacing, which leads to

$$T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}$$

The volume occupied by one molecule ($N = 1$) is of order $V = d^3$, so use the ideal gas law to get

$$d = \left(\frac{k_BT}{P}\right)^{1/3}$$

Again, use

$$T < \frac{h^2}{2mk_Bd^2}$$

but now subsitute in for d,

$$T < \frac{h^2}{2mk_B}\left(\frac{P}{k_BT}\right)^{2/3} = \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}$$

Plug in numbers for helium at atmospheric pressure. Is it quantum mechanical?

He, $m = 4m_p = 6.8 \times 10^{-27}\,\mathrm{kg}$ at one atmospheric pressure $P = 1.0 \times 10^5\mathrm{N/m^2}$, is quantum mechanical at $T < 2.8\mathrm{K}$.

Plug in numbers for hydrogen in outer space ($d = 1$ cm and $T = 3$ K). Is it quantum mechanical?

H, $m = m_p = 1.7 \times 10^{-27}\,\mathrm{kg}$ and $d = 10^{-2}\,\mathrm{m}$, is quantum mechanical at $T < 6.2 \times 10^{-14}\mathrm{K}$ and so is most definitely classical.