**PHYSICS 233: INTRODUCTION TO RELATIVITY**
**Winter 2018-2019**
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**Lecture Notes #9: Collide, Create, Annihilate**
**Thursday, February 28, 2019**

# 0 Preliminaries

**Required Reading:** *Spacetime Physics*, ch. 8

Homework 7 due Thursday, March 7:
Ch. 6: 6-1, 6-3, 6-4

Concepts for today

- Mass, energy, and momenta of SYSTEMS of particles, in contrast to properties of individual particles.
- Collisions, fission, fusion, annihilation.

# 1 Simple Systems

Most of our previous discussions of momenergy have focused on a single particle or object. Now we'll consider *systems* of particles. Is momentum conserved? Yes. Although the momentum of one particle may change, the momentum of the system is conserved. Is energy conserved? Yes. Energy may be exchanged between particles, but the total energy of the system remains same. Is mass invariant? Yes. Mass of the whole system is INVARIANT. It is also CONSERVED throughout any kind of interaction between the parts of the system. But pay attention: the mass of the system is not same as the sum of input particle masses!

Consider an elastic collision in which two glass marbles collide. We hang two glass marbles from strings like pendula that can knock together. Let marble 1 swing and it gains momentum as it swings down. Now let marble 1 hit 2. In this nearly perfect elastic collision, all of 1's momentum is given to 2. Similarly, all of 1's kinetic energy is given to 2. But the total energy and total momentum of the 1+2 system is conserved. None of this should be new to you; it's the same as in Newtonian mechanics. Momentum is conserved and energy is conserved, therefore total momenergy is the same before and after the collision.

Now examine an inelastic collision: the collision of two chewing gum globs. We simply replace the marbles with equal mass chewing gum globs (yuch - just reach under your seats!). Now pull them both back and let them collide. Just before the collision, glob 1 has momentum only in the positive $x$ direction and 2 has momentum only in the negative $x$ direction, thus the total momentum $p_i = \gamma mv - \gamma mv = 0$. After colliding, they don't move, so $p_o = 0$ also. Total momentum is conserved.

Before the collision, gum globs 1 and 2 have equal kinetic energy, thus the total energy is $E_i = 2m + 2K$. What about after the collision? Globs are not moving, so there's no kinetic energy. Does this mean that $E_o = 2m$, less than before? NO! There's no momentum, so the mass of the system, using $m^2 = E^2 - p^2$, is just the energy of the system, thus $M_{system} = E_{system}$. This system mass never changes, right? Before the collision, $E_{system} = 2m + 2K$. So, if energy is conserved, the mass of the system after collision is simply $M_{system} = 2m + 2K$.

Wait a minute! The system mass is greater than the sum of the glob masses, but neither glob is moving. Where did this extra mass come from? Well, the kinetic energy of moving globs must have been turned into heat within the stuck-together globs, or into energy of deforming the globs. The energy couldn't simply go away.

Think about it: Put the chewing gum glob system inside a sealed box, so that no energy can get in or out. How much does it weigh? Does its mass depend on whether the globs are swinging or stuck together? NO. Interactions among pieces of the closed system have no effect on the momenergy of the system. Total system momentum, total system energy, and, of course, total system mass are not changed by interactions among the constituents of the system.

Thus, conservation of energy throughout a collision implies that heat has mass, right? So, do you weight more when you're hot? You certainly feel heavier... But seriously, can one actually measure the extra mass associated with heat? Not yet. It's just too sensitive an experiment.

Take one kilogram of water at just above freezing. Heat it up to its boiling point ($\Delta T = 100K$). How much mass have you added? What's the fractional increase in the mass? Could you measure this fractional increase in mass? Remember that 1 calorie is energy to raise 1 g of water 1 degree celsius and that 1 cal = 4.184 J. We raise the temperature of 1kg, or 1000g, by 100C, thus we add $E = 10^5$ cal of energy. In Joules $E = (10^5)(4.184) = 4.184 \times 10^5$J. The mass equivalent of this heat energy is $m = E/c^2 = 4.2 \times 10^5/(3 \times 10^8)^2 = 4.7 \times 10^{-12}$kg. Thus, the fractional increase in mass is $\delta m/m = 4.7 \times 10^{-12}$. No, we cannot detect this difference in mass.

## 2 Mass of a system of particles

Let's combine some particles into a system and examine the properties of the system.

The energy of the constituents adds up to form the energy of the system. That's just conservation of energy, $E_{system} = \sum_i E_i$. Momenta of the constituents add up to form the momentum of the system. That's just conservation of momentum $\mathbf{p}_{system} = \sum_i \mathbf{p}_i$, or for each direction, $p_x = \sum_i p_x$.

But note that mass of particles does NOT add up to form the invariant mass of the system! ($m_{system} \neq \sum_i m_i$) Be very, very careful!

Let's consider another simple system of two particles: two equal masses, $m_1 = m_2 = 8$. In our laboratory, 1 moves at $v_1 = 3/5$ and 2 moves at $v_2 = -3/5$ What are the momenta? $p = \gamma m v = (1-(3/5)^2)^{-1/2}(8)(3/5) = 6$. The Lorentz factor is $\gamma = 5/4$. Thus, the particles have opposite momenta $\mathbf{p}_1 = 6, \mathbf{p}_2 = -6$.

The energy of each object obeys $E^2 = m^2 + p^2$ thus $E = \sqrt{8^2 + 6^2} = 10$. Thus, the total system energy is $E_{system} = 20$. The total momentum of the system is $p_{system} = 0$. Thus, the invariant mass of the system is $M_{system} = \sqrt{E^2_{system} - p^2_{system}} = 20$.

Wait a minute! The sum of the masses of objects is $m_1 + m_2 = 16$. But the system mass is $M_{system} = 20$. Where did the extra mass come from? What happened to the invariance of mass? The extra mass is a property of the system of masses. The relative motions of the particles, call it the "heat" of the system, is a property of that system, which cannot be observed by observing one particle at a time.

Look at the momenergy diagram of this system [see figure 8-3 on p. 224]. Add the momenergy vectors of the two masses and what do you get? Remember, $M^2_{system} = E^2_{system} - p^2_{system}$. Energies of the particles add up: $E_{system} = E_1 + E_2 = 10 + 10 = 20$. Momenta of the particles add up: $p_{system} = p_1 + p_2 = 6 - 6 = 0$. But masses of particles do NOT add to form mass of system! Mass of the system of particles is the magnitude of the momenergy vector, $M_{system} = 20$, not $8 + 8$.

## 3 Two equal masses collide

Special Relativity teaches us that Physics is the same in all inertial frames and provides tools for transforming observations from one frame to another. Therefore, we frequently choose to do our calculations in the frame that makes the problem easiest to solve. When dealing with a system of particles, it is often convenient to work in the inertial frame in which the center of mass of the system is at rest. This is because when $p_{system} = 0$, the system mass is simply the sum of the particle energies, thus $M_{system} = E_{system} = \sum_i \gamma_i m_i$.

Draw the spacetime, momenergy diagrams for each of these cases:

**Easy:** Consider two equal masses $m_1 = 8$, $m_2 = 8$ that move with velocities $v_1 = 3/5$, $v_2 = -3/5$ (thus $\gamma = 5/4$). The momenta of the masses are $p_1 = \gamma m v = (5/4)(8)(3/5) = 6$ and $p_2 = \gamma m v = (5/4)(8)(-3/5) = -6$. The energy of each object is $E^2 = m^2 + p^2$, thus each has $E = \sqrt{8^2 + 6^2} = 10$. Thus, total system energy $E_{system} = 20$. The total system momentum is 0, thus $M_{system} = \sqrt{E^2 - p^2} = E_{system} = \sum_i \gamma_i m_i = 20$.

**Harder:** Now look at the same system from a frame in which $v_2 = 0$. From this frame's point of view, our lab moves at $v_{rel} = 3/5$. Use the velocity addition formula to find $v_1' = (v_1 + v_{rel})/(1 + v_1 v_{rel}) = 15/17$. The Lorentz factor for mass 1 is $\gamma_1 = 1/\sqrt{1 - (15/17)^2} = 17/8$. The energy of 1 is $E_1' = \gamma_1 m_1 = (17/8)(8) = 17$ and its momentum is $p_1' = v_1' E_1' = 15$. Mass 2 is at rest, so it has $p' = 0$ and $E' = 8$. The system has total energy $E' = 17 + 8 = 25$ and total momentum $p' = 15 + 0 = 15$. The total system mass $M = \sqrt{(25)^2 - (15)^2} = 20$, as before. This is invariant. But does $M = \gamma m_1 + \gamma m_2$? As above $\gamma_1 m_1 = (17/8)(8) = 17$, $\gamma_2 m_2 = (1)(8) = 8$. Now $\sum_i \gamma_i m_i = 25$, not 20. That only worked in the center of mass frame.

# 4 Mass creates mass

Collide a proton with another proton. If the incoming proton has enough energy, we can create extra particles during the collision. We just have to make sure that we conserve energy, momentum, and charge (among other things - you'll learn about spin, baryon number, etc. in quantum mechanics). During the collision, create a new pair of particles: one proton and one anti-proton. An anti-proton has the same mass as a proton, but has negative charge instead of positive. If the incoming proton has just the right energy, called the "threshold energy," the four particles will stay together after the collision. With lower energy, the proton-antiproton pair could not be created.

[See Figure 8-9, p. 236]

$$p + p \longrightarrow 3p + \overline{p}$$

This time express mass and energy in units of proton mass: A proton at rest has $m = 1$, $E = 1$, $p = 0$. The incoming proton has $m = 1$, $E = 7$, $p = \sqrt{48}$. The total system energy is $E_{system} = 1 + 7 = 8$ and the total system momentum $p_{system} = 0 + \sqrt{48} = \sqrt{48}$. Therefore the total system mass $M_{system} = \sqrt{E^2 - p^2} = \sqrt{8^2 - 48} = 4$.

This collision creates a proton and an antiproton. After the collision, the particles move together with no relative velocities. The system energy, momentum, and mass are as before collision. What's the velocity of the system after the collision? Again, use $p = Ev$, thus $v = p/E = \sqrt{48}/8$.

After the collision, we have four particles, each with rest mass or rest energy $E_{rest} = 1$. The sum of the rest energies is 4, just like the total energy of the system. In this case, summing the masses adds up to the system mass because there are no internal motions.

# 5 Energy without mass: photon

What about photons? What is their mass, their energy, their momentum?

We showed how the momenergy vector is related to the spacetime displacement vector. They both point in the same direction. Now let's DRAW the spacetime and momenergy diagrams. Look at the spacetime displacement of any photon: We see that the space displacement $x$ equals time displacement $t$, so that $\tau^2 = t^2 - x^2 = 0$. What does this imply about the momenergy? The space and time components of momenergy are also equal! They have no mass, but they do have energy, right? So, if $m = 0$ and $E > 0$, what is $p$? It must satisfy $m^2 = E^2 - p^2$. Thus, for a photon, $m = 0$ and $E = p$. Recall the equality $\mathbf{p} = E\mathbf{v}$. For a photon, $v = 1$, thus $p = E$.

Therefore, photons have no mass, but they do have momentum and energy. See what happens if we collide a high-energy photon into an electron: Let's take $E_\gamma = 1.022$Mev. This is an energy typical of a gamma ray. The mass of an electron, equivalent to its rest energy is $E_e = 0.511$Mev, thus the photon has $E_\gamma = 2E_e$. Now collide this photon into an electron at rest in such a way that the photon is scattered backwards and sends the electron forwards with momentum $p = 2.4 = 12/5$. [diagram 8-6, p. 231]

We'll work in units of electron mass. Before the collision, the photon has $E_p = 2$, $p_p = 2$, $m_p = 0$ and the electron has $E_e = 1$, $p_e = 0$, $m_e = 1$. The total energy is $E = 2 + 1 = 3$. The total momentum is $p = 2 + 0 = 2$. Remember, total energy, total momentum, and system mass all remain the same throughout the collision.

After the collision, the electron has $p_e = 12/5 = 2.4$. The total input momentum was $p_{in} = 2$ and the photon reversed direction, so the outgoing photon must have momentum $p_p = -2/5 = -0.4$. The square of the energy of the photon equals the square of its momentum, so its energy must be $E_p = 2/5$. The input energy was $E_{in} = 3$, so the electron energy must be $E_e = 3 - 2/5 = 13/5 = 2.6$.

The photon flies off to the left at $v = 1$. What is the velocity of the electron? $E_e = 2.6$ in units of electron mass, $E_e = \gamma m$, where $m = 1$ in these units. $13/5 = (1 - v)^{-1/2}$. Do the algebra to solve for $v = 24/26$. OR just remember that the momenergy and spacetime displacement vectors point in same direction. The ratio of momentum/energy = 24/26, thus the ratio of space displacement/time displacement = 24/26, thus $v = 24/26$.

# 6 Photon creates mass

OK, so we accept that a photon has zero mass but does have momentum and energy. It can slam into a particle that does have mass and send it flying. Now consider something a bit more complicated: a photon can create particles!

Slam a very high-energy photon into an electron. During the collision with the electron, another pair of particles is sometimes created. Consider the case in which the the new pair includes one electron and one positron. A positron has the same mass as an electron, but has positive charge rather than negative charge.

$$\gamma + e^- \longrightarrow 2e^- + e^+$$

Now we've got a photon, two electrons, and one positron. With just the right energy for the incoming photon, sometimes the three new particles will hang together, forming a "polyelectron." Let's look at the momenergy diagram of this:

[See Figure 8-8, p. 234]

Again, we'll work in units of electron mass. The incoming photon has mass $m = 0$ and energy $E = 4$. The rest energy of the electron is its mass $m = 1$, so it has $E = 1$. Thus the total energy of the system is $E_{system} = 5$. The momentum of the photon is $p = 4$, and the electron has $p = 0$, so the input total momentum is $p_{system} = 4$. Now compute the mass of the system: $M_{system} = \sqrt{E^2_{system} - p^2_{system}} = \sqrt{5^2 - 4^2} = 3$.

After the collision: The photon is gone! All of the energy and momentum goes into the system of three massive particles. As before, $M_{system} = 3$, $E_{system} = 5$, $p_{system} = 4$. What is the velocity of this system? Again, use the fact that the momenergy vector follows the spacetime displacement vector: Momentum/energy = 4/5, thus space/time displacement = 4/5 and $v = 4/5$.

# 7 Converting mass to energy: fission, fusion, annihilation

The fact that "system" mass differs from the sum of masses of individual particles in the system is what allows us to get energy out of fission and fusion. Fission is splitting of atomic nuclei into lighter nuclei. Fusion is the process of combining light nuclei into heavier.

Wait - sometimes, SPLITTING an atomic nucleus yields energy. Other times, FUSING nuclei into a heavier one yields energy. How can both processes yield energy?

Critical to understanding fission and fusion is the idea of "mass per nucleon." We've seen how systems of particles can have masses that differ from the sums of their masses. Likewise, collections of protons and neutrons in the nuclei of atoms have different average mass per particle, depending on how many neutrons and protons are in the nucleus. This is because the sub-atomic forces holding the nucleus together yield different internal "system" energies depending on the configuration of nucleons.

Draw the diagram of mass per nucleon. [See Figure 8-9, p. 238]

The most stable nucleus, the element with the smallest mass per nucleon, is Iron. Why is "smallest mass per nucleon" the same as "most stable?" It means that we have to ADD the most energy per nucleon to pry the neutrons and protons free. The total "system" mass of a nucleus is LESS than the sum of masses of the same number of "free" neutrons and protons. So the most stable nucleus is one that requires the most energy per nucleon to break it into its consituent particles.

Recall notation for elements: upper number is nucleons (protons plus neutrons), lower is number of protons.

**Fission**

$$n + \mathrm{U}_{92}^{235} \longrightarrow \mathrm{U}_{92}^{236} \longrightarrow \mathrm{Rb}_{37}^{95} + \mathrm{Cs}_{55}^{141} + \mathrm{energy\,(photons)}$$

Slam a neutron into a $\mathrm{U}_{92}^{235}$ nucleus, producing (momentarily) $\mathrm{U}_{92}^{236}$ which decays into $\mathrm{Rb}_{37}^{95}$ and $\mathrm{Cs}_{55}^{141}$. Before, during, and after this collision, there are 92 protons and 144 neutrons. But "useful" energy comes out of this reaction, because the Rb and Cs nuclei are more tightly bound together than the original U nucleus. And so on towards Iron (Fe).

**Fusion**

$$2\mathrm{D}_1^2 \longrightarrow \mathrm{He}_2^4 + \mathrm{energy\,(photons)}$$

Combine two deuterium nuclei $D_1^2$ into one helium nucleus $He_2^4$. Deuterium is "heavy" hydrogen, with an extra neutron. So-called "heavy water" is $D_20$ rather than $H_2O$. Helium nucleus is more bound, has smaller mass per nucleon, than deuterium, so we get energy out of this reaction.

### Annihilation

$$e^- + e^+ \longrightarrow 2\gamma$$

Put a positron and an electron in slow orbit around one another. Eventually they annihilate and out come 2 or 3 photons! All of the mass is consumed, leaving only energy in photons.

In units of electron mass: Before the annihilation: the electron and positron each have $E = m = 1$, $p = 0$. The total system mass is $M_{system} = 2$.

After the annihilation: 2 photons with opposite momentum $p = +1, -1$ fly out. The total momentum is $p_{system} = 0$, as before. Note that conservation of momentum REQUIRES that at least 2 photons come out, otherwise it is impossible to balance the outbound momenta. Each photon has energy $E = 1$. The total energy remains $E_{system} = 2$ and the sum of photon masses is $m = 0$. Wait a second, isn't mass invariant? Yes, that means that mass is same in every frame. But remember that mass of system of particles is NOT the sum of individual masses. The total system mass $M_{system} = \sqrt{E^2 - p^2} = \sqrt{2^2 - 0} = 2$, as before.

Each photon has $m^2 = E^2 - p^2 = 0$, but the SYSTEM has non-zero mass! In other words, a collection of photons can have mass, even though none of the photon has mass. Think about a box full of electrons and positrons. Weigh each of them, add up the masses. Now put them in the box, seal it up and insulate it so that no heat can escape. Wait until they have annihilated each other. Weigh the box again - same weight! Let the photons out one by one. None of them have mass, but when you're done, the box is lighter!