PHYSICS 233: INTRODUCTION TO RELATIVITY
Winter 2018-2019
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Lecture Notes #8: Momenergy
Thursday, February 21, 2019

0 Preliminaries

Required Reading: *Spacetime Physics*, ch. 7

Homework:
HW6 is due Thursday, February 28
Chapter 5, problems 5-2, 5-4, 5-6.

The goal of this section of the course is to understand the interactions of particles and photons. Events of interest include *collision, creation* and *annihilation* of particles.

Concepts for today:

- Momenergy = momentum and energy, unified in SR just like spacetime = space and time, momentum and energy are unified in SR
- 4-vectors
- Conservation of momenergy
- Invariant of momenergy is mass, $E = mc^2$

1 Newtonian energy and momentum

You're probably familiar with energy and momentum from Newtonian mechanics: Kinetic energy of a particle is $E = mv^2/2$. Momentum of a particle is $p = mv$. Here v is in "conventional" units of meters per second. Energy is in Joules, while momentum is in kilogram meters per second. Because they depend on the velocity, which depends on the frame of the observer and which cannot exceed the speed of light, you already suspect that these quantities will look different in SR. That is, the Newtonian expressions are just the low-velocity limits of the more exact SR quantities. We'll show that energy and momentum are related to each other in the same way that time and space are related.

In Newtonian mechanics, energy is conserved and momentum is conserved. In a collision between particles, the particles may exchange energy, they may exchange momentum, but the total energy is conserved and the total momentum is conserved. What is conserved in SR? That is, what is the invariant quantity?

In spacetime, we found that space and time depend on the reference frame, but that the spacetime interval $t^2 - x^2$ is invariant. For momenergy, we'll find that the invariant is the rest mass, $m^2 = E^2 - p^2$.

Wait a minute. That equation makes no sense. The units are all wrong! Mass in kilograms, energy in Joules (kilogram meters squared per second squared), and momentum in kilogram meters per second. This is the same problem as in spacetime – we just need to convert to spacetime units, in which we measure space and time in the same units. With time and distance in meters, $c = 1$ and all velocities are dimensionless.

Here are our new units, using $c = 1$: Energy $E = mv^2/2$ has units of kilograms. Momentum $p = mv$ has units of kilograms. So, mass in kilograms, Energy in kilograms, and momentum in kilograms. Strange, but no stranger than light-meters, right?

2 4-vectors and the momenergy arrow

Recall that a vector has both a magnitude and a direction. For example, a velocity in 2-dimensional space has components v_x, v_y. It has magnitude $v = \sqrt{v_x^2 + v_y^2}$ and a direction with respect to the x axis $\theta = \tan^{-1}(v_y/v_x)$. Likewise, a 3-vector of velocity in Euclidean space would have components v_x, v_y, v_z. Or simply a 3-vector of position x, y, z.

What about vectors in spacetime? The 4-dimensional spacetime displacement is a 4-vector, with components t, x, y, z. What is the magnitude of this vector? The invariant spacetime interval, which is the same as the proper time

$$\tau = \sqrt{t^2 - x^2 - y^2 - z^2}$$

Thus, in Newtonian mechanics, quantities like velocity are described by 3-vectors. In SR, we need 4-vectors because of the way space and time are intertwined.

In Newtonian mechanics, the momentum and energy of a particle depend on its mass and its 3-vector velocity: $E = mv^2/2$, where v is the magnitude of the velocity 3-vector. $\mathbf{p} = m\mathbf{v}$, where $\mathbf{p}$ is a vector quanitity - it has both a magnitude and direction, which depends on the 3-vector $\mathbf{v}$. The 3-vector velocity depends on the spatial displacement and the time, e.g., $v_x = \Delta x/\Delta t$

In SR, energy and momentum are unified into momenergy, which also depends on the mass of the particle. But, instead of using the 3-vector velocity, it now depends on the

4-vector spacetime displacement and the proper time.

The momenergy 4-vector is
(**momenergy**) = (mass) × (**spacetime displacement**) / (proper time)
where both **momenergy** and the **spacetime displacement** are 4-vectors. Mass here means the rest mass of a particle. It's an intrinsic property of matter, which doesn't change with rest frame.

The momenergy 4-vector points in the same direction as the spacetime displacement. Where else would it point? If the particle moves along a path from event 1 to event 2, its momenergy vector must point along that same path.

REMEMBER that (**spacetime displacement**) is a 4-vector, while (proper time) is the magnitude of that vector. The 4-vector is a geometric object, including both a direction and magnitude. The spacetime displacement 4-vector has components t, x, y, z. The proper time is just the magnitude.

In our simplified spacetime, with only one spatial dimension, we see that the energy component of momenergy points along the time axis. The momentum component points in the spatial direction. The magnitude of the momenergy vector is just like the spacetime interval. Recall spacetime interval $t^2 - x^2$. Likewise, the square of the momenergy magnitude is the time part squared minus the space part squared, $E^2 - p^2$.

OK, now let's look in detail at the components of the momenergy 4-vector:
Time component = Energy = (mass) × (time displacement) / (proper time).
Recall our notation: t for time displacement (which depends on frame) and τ for proper time. Thus, in calculus notation,

$$E = m\frac{dt}{d\tau}$$

Look at one space component of momenergy, in x direction:
Space component = momentum = (mass) × (x displacement) / (proper time).
Again, in calculus notation, the x component of momentum is

$$p_x = m\frac{dx}{d\tau}$$

Likewise, $p_y = mdy/d\tau$ and $p_y = mdz/d\tau$

Does this look familiar? Almost. Newtonian momentum is $p = mv = mdx/dt$. Here we've just replaced the time, which depends on reference frame, with proper time, which is independent of reference frame.

We'll return in a moment to look at the energy and momentum components in more detail. But first, let's look at the magnitude of the momenergy 4-vector. As with the spacetime interval, take the square of the time component minus the squares of the space components:

$$E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 \frac{[(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2]}{(d\tau)^2}$$

But what is $(d\tau)^2$? It's just the square of the infinitesimal proper time,

$$(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

thus

$$E^2 - p^2 = m^2 \frac{(d\tau)^2}{(d\tau)^2} = m^2$$

Note that $\mathbf{p}$ is a 3-vector, with magnitude p.

So, the invariant quantity of momenergy is just the mass! The time component, energy, and spatial components, the momenta, vary with reference frame, but the magnitude of the momenergy vector, the mass of the particle, remains the same.

What happens when we set a particle in motion? At rest, its worldline just follows the time axis. Likewise, it's momenergy is purely in the time direction – it has energy but no momentum. Now, give it a kick in the x direction. It gains momentum, so its momenergy now has a spatial component – momentum in the x direction.

What happens to the energy? Recall $E = m(dt/d\tau)$. The proper time, which is the clock time in the particle's frame, is no longer the same as laboratory time. Lab time can only be longer than proper time, thus E is larger than before.

What about the magnitude of the momenergy, defined by $m^2 = E^2 - p^2$? Remarkably, it stays the same. The square of the energy gets larger by exactly the same amount as the momentum squared. The momenergy magnitude, the mass of the particle, is the same regardless of what speed it moves at! But the energy itself does change:

$$E^2 = m^2 + p^2$$

thus

$$E = \sqrt{m^2 + p^2}$$

(Recall $t^2 = \tau^2 + x^2$.)

What happens when we transform to another frame of reference? Again, start with a particle at rest, that is, with $p = 0$. Momenergy points along the time axis. Transform to another reference frame, in which the particle is in motion with some velocity. Now the particle has different energy and momentum, but the momenergy magnitude

$$m^2 = E^2 - p^2$$

is *invariant*. We can now make a simplified "momenergy diagram" like our spacetime diagrams. (DRAW the invariant hyperbola for momenergy, with energy along the "time" axis and momentum along the "space" axis.)

3 Momentum: the space part of momenergy

Let's compare again the Newtonian and SR versions of momentum.
Newton: $p_x = mdx/dt$, likewise for p_y, p_z.
SR: $p_x = mdx/d\tau$.
What's the difference? Proper time τ is the spacetime distance between events on the worldline of the particle. This is the time as measured in frame of the moving particle, in which there is no spatial motion of the particle.

The invariance of spacetime interval, comparing particle's frame with any other implies

$$(d\tau)^2 = (dt)^2 - (dx)^2 = (dt)^2 - (vdt)^2 = (1 - v^2)(dt)^2$$

Thus,

$$d\tau = dt(1 - v^2)^{1/2} = \frac{dt}{\gamma}$$

With this relation for $dt/d\tau$,

$$E = m\frac{dt}{d\tau} = m\gamma$$

and

$$p_x = m\frac{dx}{d\tau} = m\frac{dx}{(dt/\gamma)} = m\frac{dx}{dt}\gamma = mv_x\gamma$$

Likewise for p_y, p_z.

At low velocity, $\gamma \approx 1$ and we recover the Newtonian equations for energy and momentum. Well, almost. What about units? In our SR equations, velocity is dimensionless, as a fraction of the speed of light. Thus, p_x is units of kilograms. To get back to conventional units, just multiply by c in conventional units of meters per second, $p_{conv} = pc$

For energy, multiply by the square of the speed of light, so that energy in conventional units is $E_{conv} = Ec^2$. Fully, $E_{conv} = m\gamma c^2$. At rest, $\gamma = 1$, thus the famous equation

$$E_{conv} = mc^2$$

4 Review of Momenergy (so far)

E, p_x, p_y, p_z are components of momenergy just like t, x, y, z are components of spacetime. In both cases, these are components of a 4-vector that has both a magnitude and a direction.

Invariant of momenergy is mass $m^2 = E^2 - p^2$, similar to the invariant spacetime interval or proper time $\tau^2 = t^2 - x^2$ (for 1D case). The mass m is the invariant magnitude of the momenergy 4-vector. The proper time τ is the invariant length of a spacetime displacement 4-vector.

Units: remember that we're working in units where time is measured in light-meters, so velocities are dimensionless. Thus, energy, momentum, and mass all have the same units of kilograms.

The momenergy 4-vector is related to the spacetime displacement 4-vector:
(Momenergy 4-vector) = (mass)(spacetime displacement 4-vector)/(proper time)
Thus, $E = mdt/d\tau$ $p_x = mdx/d\tau$, etc.

The 4-vector of momenergy points in the same direction as the spacetime displacement. For a particle at rest, the spacetime displacement between events only has a time component, no space component. Thus, the momenergy has only an energy component, no momentum. Set the particle in motion, it increases in both energy and momentum, but the magnitude of momenergy remains the same, it's just the mass of the particle $m^2 = E^2 - p^2$.

Momentum part, $p_x = mdt/d\tau = \gamma m v_x$. At low velocity, $\gamma = 1$, so $p_x = mv$, just like in Newtonian mechanics.

To get from SR units of momentum, we need to multiply the momentum by the speed of light, $p_{conv} = pc$.

5 Energy: the time part of momenergy

Again, $E = mdt/d\tau = m/(1 - v^2)^{1/2} = m\gamma$.

Compare with the Newtonian kinetic energy $E_{Kinetic} = K_{Newton} = mv^2/2$. When $v = 0$, $K_{Newton} = 0$. But, in SR, when $v = 0$, $E = m$. This is the *rest energy* of the particle, equivalent to its *rest mass*. The rest energy obviously does not go to zero at $v = 0$.

How can we reconcile the Newtonian and special relativistic descriptions of energy? Simple. The SR energy is the total energy of the particle, whereas the Newtonian equation includes only the kinetic energy at low speed. We simply need to define kinetic energy as energy in addition to the rest energy, $E = m + K$ or

$$K = E - m = m[\gamma - 1]$$

At $v = 0$, $\gamma = 1$, thus $K = 0$ in agreement with the Newtonian kinetic energy. Well, almost. Again we have a problem of units. Here we have kinetic energy in kilograms, not Joules. As we discussed before, we multiply by c^2 to get back to conventional units.

Now, let's work in conventional units and check that we recover the Newtonian kinetic energy as v approaches zero:
In conventional units,

$$K = (E - m)c^2 = mc^2 \left[\frac{1}{\sqrt{1 - (v/c)^2}} - 1 \right]$$

Look at the term $1/(1 - (v/c)^2)^{1/2}$. For small v/c, you can show that

$$1/(1 - (v/c)^2)^{1/2} \approx 1 + (v/c)^2/2$$

Now,

$$K = mc^2[1 + (v/c)^2/2 - 1] = mv^2/2$$

as required so that we recover the Newtonian energy at low velocity.

Again, in conventional units, the rest energy of a particle is $E = mc^2$. Its total energy, $E = \gamma mc^2$, increases with velocity. The kinetic energy part of this is just the total minus the rest energy, $K = E - E_{rest} = mc^2(\gamma - 1)$. Back to our $c = 1$ units, $E_{rest} = m$, $E = \gamma m$, and $K = m(\gamma - 1)$.

6 Relation between Energy and momentum

Note that $p = \gamma mv$ and $E = \gamma m$. Thus

$$p = vE$$

Surprised? You shouldn't be. Look at the spacetime diagram. In a spacetime diagram, a particle at constant velocity has $x = vt$. The momenergy 4-vector always points in the same direction, so it matches the tilt of the worldline.

In detail, using simple calculus:

$$E = m\frac{dt}{d\tau}$$

and

$$p = m\frac{dx}{d\tau}$$

thus

$$\frac{p}{E} = \frac{m\frac{dx}{d\tau}}{m\frac{dt}{d\tau}} = \frac{dx}{dt} = v$$

7 Conservation of momenergy

In Newtonian mechanics, we had conservation of energy and conservation of momementum. Smash some particles into each other,

1. Conservation of energy $\sum E_{in} = \sum E_{out}$
2. Conservation of momentum $\sum \mathbf{p}_{in} = \sum \mathbf{p}_{out}$ where you have to remember that $\mathbf{p}$ is a vector quantity, which can be broken down into components, $\sum p_{x,in} = \sum p_{x,out}$.

What about in SR? What is conserved? Regardless of which free-float frame from which we observe the collision, the total momenergy is conserved! *In a particular free-float frame*, the time and space parts of momenergy are individually conserved. In other words, we now have conservation equations that look the same as before. Within an inertial frame we have the following conservation relations:

1. Conservation of time part of momenergy $\sum E_{in} = \sum E_{out}$ and
2. Conservation of space part of momenergy $\sum \mathbf{p}_{in} = \sum \mathbf{p}_{out}$. As above, this vector equality can be broken into components.

Now, we need to be very careful in our terminology:

INVARIANT (with respect to reference frame) = quantity that has the same value in any reference frame. Example: spacetime interval, magnitude of momenergy

CONSERVED (with respect to event such as collision) = quantity that remains unchanged through some interaction, but the actual value of this quantity could depend on reference frame. Example: Energy of set of particles, momentum of set of particles

CONSTANT (with respect to time or place) = quantity that does not change with time. Example: speed of light, mass of electron

These are not mutually exclusive; some quantities can be described by more than one of these. For example, speed of light is invariant, conserved, and constant. Magnitude of momenergy is invariant and conserved.

8 Summary

(momenergy 4-vector) = (mass) (spacetime displacement 4-vector) / (proper time for that displacement)

Time and space components of momenergy E, p_x, p_y, p_z, respectively, behave a lot like time and space components of spacetime interval.

Invariant magnitude of momenergy, the rest mass

$$m^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

or just

$$m^2 = E^2 - p^2 = E'^2 - p'^2$$

where p is the magnitude of the momentum 3-vector.

Space components

$$p_x = m\frac{dx}{d\tau} = mv_x\gamma$$

Likewise for p_y, p_z.

Time component

$$E = m\frac{dt}{d\tau} = m\gamma$$

At rest, $v = 0$, thus $E_{rest} = m$.

Kinetic energy is total energy minus rest energy,

$$K = E - m = m(\gamma - 1)$$

9 Sample problem 7-3

Write down the four components of the momenergy 4-vector in the given frame in the form $\{E, p_x, p_y, p_z\}$.

(a) Particle moves in positive x direction with kinetic energy equal to three times its rest energy ($K = 3m$).

Rest energy is just the mass m. Total energy is kinetic plus rest energy, thus

$$E = 3m + m = 4m$$

Momenta in y and z are zero. Get momentum in x direction from invariance $m^2 = E^2 - p^2$, thus

$$p^2 = E^2 - m^2 = (4m)^2 - m^2 = 15m^2$$

$$\{E, p_x, p_y, p_z\} = \{4m, (15)^{1/2}m, 0, 0\}$$

(b) Same particle is observed in a frame in which kinetic energy equals its mass ($K = m$).

Total energy is kinetic plus rest energy, thus

$$E = m + m = 2m$$

Still zero momentum in y or z directions. Using invariance, get x momentum,

$$p^2 = E^2 - m^2 = (2m)^2 - m^2 = 3m^2$$

$$\{E, p_x, p_y, p_z\} = \{2m, (3)^{1/2}m, 0, 0\}$$

(e) A particle moves with equal x, y, z momentum components ($p_x = p_y = p_z$) and with kinetic energy equal to four times its rest energy ($K = 4m$).

Total energy equals kinetic plus rest

$$E = 4m + m = 5m$$

Momentum, $p_x = p_y = p_z$ and $p^2 = p_x^2 + p_y^2 + p_z^2 = E^2 - m^2$ Same as $3p_x^2 = (5m)^2 - m^2 = 24m^2$,

$$p_x = 8^{1/2}m$$

$$\{E, p_x, p_y, p_z\} = \{5m, 8^{1/2}m, 8^{1/2}m, 8^{1/2}m\}$$