INTRODUCTION TO RELATIVITY
Winter 2018-2019
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Lecture Notes #7: Spacetime
Thursday, February 14, 2019

0 Preliminaries

Required Reading: *Spacetime Physics*, chs. 5 and 6

Next homework: HW5 will be due Thursday, February 21. Chapter 4: Problem 4-1 (Just one problem! Explanations, please. The answers are in the book.)

1 Worldlines

The motion of a particle through spacetime defines its "worldline" Let's look at worldlines of particles moving at constant velocities.

[see diagram on p. 145]

First look at just the spatial dimension. All particles start at $x = 0$ and move with some velocity v_i. Now draw their paths, we call them worldlines, in the spacetime diagram. A larger angle with respect to time direction means larger velocity. Light speed is 45 degree angle – no worldline can ever have an angle greater than 45 wrt time axis. All these objects move at constant velocity, so their worldlines are straight. What happens when an object changes direction? We get curves in its worldline.

An example: a train on a straight track that accelerates and decelerates. Since I take the train to and from Drexel everyday, which is the largest distance that I travel on a regular basis, you could say this is the worldline of my life!

The train starts at rest, moving vertically in spacetime diagram. The Worldline curves toward positive x as it accelerates. Then a straight line at constant velocity. Curves back as it decelerates, moves straight upward at rest.

As much as I would like SEPTA to buy some faster-than-light trains, unfortunately the worldline will never be anywhere close to 45 deg from vertical. NO worldline can ever be more than 45 deg from vertical.

2 Length along paths in spacetime

What's the shortest distance between two points? A straight line, right? Not in spacetime!

In Euclidean geometry, two locations separated by Δx and Δy have total separation $d^2 = (\Delta x)^2 + (\Delta y)^2$. To get from A to B, the shortest path is along a straight line connecting these points. Any other path, say, one that curves, is longer.

[Draw the path in space as on p. 148]

Strangely, in spacetime, a curved path is shorter than a straight one. Look at how we measure distance in spacetime, $d^2 = (\Delta t)^2 - (\Delta x)^2$ The more we move around in space, i.e., the larger Δx is, the shorter is the total interval! (For fixed Δt, of course.) Just compare the length in spacetime for a straight path, moving upwards with no side to side motion, to the length for curved path.

What does this mean for the length of proper time between events? Remember, proper time is the time measured by the watch of an observer that moves along the path, i.e., the amount that he ages along the path. Strangely, if he follows a curved path from A to B he ages less. The maximum ageing occurs if he follows a straight line in spacetime between the events!

The spacetime interval between events along the worldline of an observer in a frame that moves along a curved world line is

$$t'^2 - x'^2 = t'^2 - (0)^2 = t^2_{proper}$$

For any other observer, any pair of events along the worldline has

$$t^2 - x^2 = t^2_{proper}, t^2 = t^2_{proper} + x^2$$

Thus, $t > t_{proper}$.

Note that, as the space interval approaches the time interval ($v \to 1$), the proper time interval approaches zero. For a photon, no proper time elapses!

3 Kinked Worldline

[See figure on p. 153]

Compare three paths from O to B, that go through events P, Q, R and compute the total proper time along each path.

OPB $t^2_{proper} = t^2 - x^2$ $x = 0$ between O and B, thus $t_{proper} = 10\,\mathrm{m}$

OQB Look at part of path, just OQ. $t^2_{proper} = (5\,\mathrm{m})^2 - (4\,\mathrm{m})^2 = 9\,\mathrm{m}^2$ Total proper time is twice this, $t_{proper} = 6\,\mathrm{m}$ Again, clearly, the kinked worldline has smaller proper time. Now for a tricky case:

ORB Part of path, just OP $t^2_{proper} = (5\,\mathrm{m})^2 - (5\,\mathrm{m})^2 = 0$ Total proper time is twice this, $t_{proper} = 0$

What, NO proper time elapsed? Does this mean that an observer in this frame does not age at all? CORRECT! Look carefully at this world line: the velocity is $v = x/t = 5\,\mathrm{m}/5\,\mathrm{m} = 1$. No material object can travel at $v = c$, so there's no problem – there is no clock that can travel at this speed, no process that "stops running." But massive particles – objects – can get arbitrarily close to $v = c$, and so the proper time can get very close to zero, but never exactly zero.

Note that these times and distances are as measured in a specific free-fall frame, but the spacetime interval $t^2 - x^2$ is the same in every free-fall frame, so we'd get the same result if we used observations from a different frame.

What about observations from the frame moving at $v = c$? Nonsense – there can be no "observations" from a frame that includes no massive objects. No clock, no rod, no measuring device at all can travel at this speed, so there can be no such "observations."

4 Stretch Factor: Ratio of Frame-clock to Wrist-watch time

[SKIP this if we're short on time - get it?]

Imagine a clock sitting at rest with a light on top. Imagine that the clock reads meters of time rather than seconds. Let the light flash at $t' = 0\,\mathrm{m}$, then again at $t' = 6\,\mathrm{m}$. What is the proper time between the flashes? Obviously, it's $\Delta t' = 6\,\mathrm{m}$, since that's what a clock sitting at both events reads.

Now place that clock in a rocket and observe the flashes from a "laboratory" frame in which the rocket flies past at $v = 0.8$. How much time elapses between the flashes, as observed in the lab frame? Look at spacetime interval:

$$t'^2 - x'^2 = t^2_{proper} = t^2 - x^2$$

But now the events are separated in space. By how much? by $x = vt$, where v is the velocity of the rocket. Now,

$$t^2_{proper} = t^2 - (vt)^2 = t^2(1 - v^2)$$

Thus,

$$t^2/t_{proper}^2 = 1/(1 - v^2)$$

so the stretch factor between proper time and lab time, or time measured in any frame other than the rocket is $\gamma = 1/\sqrt{1 - v^2}$

This is the ratio of "frame time," the time between events as measured in a particular free-fall frame, to the "proper time," the time as measured by a clock that moves along the worldline connecting the events.

5 Problem 5-8: The Twin Paradox Resolved - Once and For All

The key to the twin paradox is simultaneity, or lack thereof.

Let's draw the spacetime diagram of the astronaut's trip to Canopus and back. Recall: An astronaut travels at $v = 99/101$, arriving at Canopus at $t' = 20\,\mathrm{y}$ as measured in her frame, but $t = 101\,\mathrm{y}$ later as measured in an Earth-linked frame. She turns around at Canopus and returns home at $v = 99/101$, arriving home at $t' = 40\,\mathrm{y}$, as measured on her clock and $t = 202\,\mathrm{y}$ as measured on Earth clock.

[See diagram from p. 169]

Draw frame of spacetime diagram. Earth worldline is simply the t axis.

O is launch from Earth at $t = 0, x = 0$.

T is turnaround at Canopus at $t = 101, x = 99$.

C is return to Earth, at $t = 202, x = 0$.

Now, let's think about simultaneity of events in different references frames: In the Earth frame, simultaneous events are on same horizontal line in this diagram.

What events are simultaneous in the frame of the rocket on its way to Canopus? Use the inverse Lorentz transformation:

$$t' = -v_{rel}\gamma x + \gamma t$$

where $v_{rel} = 99/101$ and $\gamma = 101/20$ (we know this because that's the ratio of time measured on Earth to time measured by astronaut).

We find the line of simultaneity that includes the turnaround event T, which is $t' = 20\,\mathrm{y}$. Thus

$$20\,\mathrm{y} = -(99/101)(101/20)x + (101/20)t$$

Multiply through by $(20/101)$

$$400/101 = -(99/101)x + t$$

or

$$t = 0.980x + 3.96$$

[Draw this line.] This is the line along which events in the rocket frame are simultaneous with the rocket's arrival at Canopus. At $x = 0$, $t = 3.96\,\mathrm{y}$, so this means that Earth time of $t = 3.96\,\mathrm{y}$ is simultaneous with arrival at Canopus.

What events are simultaneous in the frame of the rocket on its way back from Canopus? Now the astronaut turns the rocket around, thus jumping from an outgoing reference frame to an incoming reference frame. This reverses v_{rel} in the equation above which reverses the slope of the line of simultaneity for the astronaut. Events B and T both lie on this line. By symmetry, event B is at Earth time $t = 202 - 3.96 = 198.04$.

Not sure about this? We can derive the equation for the second line, using the same math as for the first. But we need to pick a different reference event to set the zero of space and time in both the Earth and rocket frames. Use the turnaround T as the new zero point. (Note that this occurs at $x = 99, t = 101$ in the original Earth coordinates.) Now use the inverse Lorentz transformation, as above, to find the rocket frame's line of simultaneity with the turnaround event:

$$t' = -v_{rel}\gamma x + \gamma t$$

In these new coordinates, T occurs at $t' = 0$, thus

$$0 = -(-99/101)(101/20)x + (101/20)t$$

which yields $t = -0.980x$. Simultaneous with the turnaround T is an event at Earth at which Earth clocks read, at $x = -99$, $t = 97.04\,\mathrm{y}$.

Remember that this is offset relative to our original coordinates by $t = 101\,\mathrm{y}$, thus the Earth clock time in the old coordinates is simply $101\,\mathrm{y} + 97.04\,\mathrm{y} = 198.04\,\mathrm{y}$, again at event B.

Let's go over the trip again: In each frame, let there be a string of observers who travel in that frame with the astronaut, spread along a line that stretches off behind Earth and off into the distance beyond Canopus. During the outward journey, as each outgoing-frame observer passes Earth, he notes the time on the Earth clocks. If they compare notes, they deduce that Earth clocks run slow relative to outgoing-frame clocks. Note that, at turnaround T, when $t' = 20\,\mathrm{y}$ in the rocket frame, they observe Earth clocks to read only $t = 3.96\,\mathrm{y}$. Just look at the line of simultaneity. The time interval during their outward journey is just time segment OA on Earth clocks.

Now, the astronaut jumps from the outgoing frame to the incoming frame. He joins a frame with a new group of incoming-frame observers. Simultaneous with his joining their frame, they observe Earth clocks to read $t = 198.04\,\mathrm{y}$. Again, look at the line of simultaneity. As the incoming frame moves towards Earth, observers in that frame look at the Earth clock as they pass by. When they compare notes, they deduce that Earth clocks run slow relative to incoming-frame clocks. On incoming-frame clocks, it takes $t' = 20\,\mathrm{y}$ for the astronaut to get back to Earth, while they see only $t = 3.96\,\mathrm{y}$ elapse on the Earth clocks. The time interval for the inbound journey is just the time segment BC in Earth-clock time.

OK, now we've accounted for time OA and BC in Earth-frame time. What about the huge interval AB? Where did that go? On the way out, all of the events on line AB are in the *future* of the astronaut, since AT is a line of simultaneity for the outbound rocket frame. On the way back, all the events on line AB are in the *past* of the astronaut, since BT is a line of simultaneity for the incoming frame.

What happened? As the astronaut turned around, "jumping" from the outbound to inbound frames, her line of simultaneity swept from AT to BT. If she had simply stopped at Canopus and stayed, her line of simultaneity would be horizontal, just like the Earth frame's.

Think about the passage of time on Earth as a long series of clock tick events. On the outbound journey, there are fewer Earth clock ticks than rocket frame ticks. Likewise, on the inbound journey, there are fewer Earth clock ticks than rocket frame ticks. But, as she turned around, a whole pile of clock tick events moved from her future to her past!

Tuesday, February 19, 2019

0 Preliminaries

Required Reading: *Spacetime Physics*, chs. 5 and 6

Next homework:

HW5 is due Thursday, February 21. Chapter 4: Problem 4-1 (Just one problem! Explanations, please. The answers are in the book.)

HW6 will be due on Thursday, February 28.
Chapter 5: problems 5-2, 5-4, 5-6.

Concepts for today: Review classification of intervals between events:

- Timelike
- Spacelike
- Lightlike

Light cone that defines past and future of an event

1 Relations between events

In Euclidean geometry, the square of the distance between events is always positive or zero

$$d^2 = x^2 + y^2 + z^2$$

It's just the square of the separation in space.

In spacetime, the square of the interval behaves differently, because of the opposite sign of the time and space contributions to the spacetime interval.

1.1 Timelike interval

Spacetime interval is dominated by separation in time. Thus,

$$t^2 - x^2 > 0$$

Look at the worldline of a particle or, for that matter, a person. (DRAW our curving worldline including acceleration, deceleration, etc.) This is spacetime as viewed in a stationary laboratory frame, from which we are observed to accelerated, decelerate, etc. As we move through spacetime, we always travel at less than the speed of light.

Look at any two events along our worldline, as seen by the laboratory. If they are separated by time t, then in units of meters and light-meters, we know that $x = vt < t$ because $v < 1$. Thus, $t^2 - x^2 > 0$.

In our own inertial frame, there is no spatial separation, and the interval is just the time separation. Recall that this is the PROPER TIME between events,

$$\tau^2 = t^2 - x^2$$

and again note that this implies that the time separation as viewed from any other frame is longer than in our frame

$$t^2 = \tau^2 + x^2$$

Consider the spacetime diagrams of two events, viewed from three different frames [See diagram on p. 173].

In laboratory frame, a flash bulb sits on a table. At $t_A = 0$ it flashes, some time t_B later it flashes again.

Now look at the same event from a rocket that flies through the lab in the positive x direction. The second event occurs at $x' < 0$, and at time $t'_B > t_B$. Transforming the two events from the lab to right-moving rocket frame, we find that the invariance of the spacetime interval $t^2 - x^2$ lands event B on the invariant hyperbola.

Likewise, the same event viewed from left-moving rocket frame lands event B on same hyperbola, this time with $x'' > 0$ and $t''_B > t_B$.

This is what the invariant hyperbola looks like for all timelike intervals.

1.2 Spacelike interval

Two events that are separated by more distance than time. Thus

$$t^2 - x^2 < 0$$

Since we don't want to have spacetime intervals whose square is negative, we write the spacelike interval as

$$s^2 = x^2 - t^2$$

An obvious case of a spacelike interval is that between two events that occur *simultaneously* in one frame. They occur at the same time, but at different locations, so obviously $t^2 - x^2 < 0$, but $s^2 = x^2 - t^2 = x^2 > 0$. This interval between events that occur simultaneously in a frame we call the PROPER DISTANCE,

$$s^2 = x^2 - t^2$$

This definses another "invariant hyperbola" in spacetime. Note that, in any other frame, the distance between two events will be larger than the proper distance,

$$x^2 = s^2 + t^2$$

Length expansion? No. This is not a measurement of distance between two points, which an observer would make simultaneously in his frame. Those events are not simultaneous in the other frame! It's the lack of simulataneity that makes x larger. To see what another frame measures as a length, we need to examine its line of simultaneity.

Look at what happens when we transform two events with a spacelike separation from one frame to another [See diagram on p. 174].

In the lab frame we place two flash bulbs on a table, separated by a few meters. Midway between them we wire up a switch that will set them both off simultaneously in our frame, so they both flash at $t = 0$ in our frame, but are separated by distance x.

From a rocket frame that moves in the positive x direction, the flash further to the right is observed to occur BEFORE the first $t' < 0$ and at a point further away $x' > x$ in the moving frame. Check that

$$t' = -v_{rel}\gamma x + \gamma t$$

So positive x implies negative t'.

The invariance of the interval $t^2 - x^2 = t'^2 - x'^2$ yields, for $t = 0$, $x'^2 - t'^2 = x^2$. So the transformed events land the rightmost flash somewhere on an invariant hyperbola that curves above and below the space axis. This is a different hyperbola than the one for timelike events!

Likewise, as viewed from a rocket that moves to the left in the laboratory frame, the rightmost flash occurs after the left one, with $t'' > 0$ and $x'' > x$.

1.3 Causality and spacelike intervals

Look carefully at the spacelike hyperbola. This shows the possible locations of the second event for any possible velocity. Draw a line between the first event, at $t = 0$, $x = 0$, and the second event, which lies somewhere along that hyperbola. There is always more space than time separation between those two events.

Can a signal be transmitted between those two events? NO. That would require a signal to travel at faster than the speed of light. Since no signal can travel between them, they cannot influence each other, no matter what reference frame we observe them in.

Thus we say that these events are not CAUSALLY CONNECTED. They cannot influence each other.

1.4 Lightlike intervals

Consider two events that are separated by light signals. In the laboratory frame, I instruct my assistant to flash his light when he sees my light flash. What's the spacetime separation? The time between the events is $t = x/v = x$ because light travels at $v = 1$ in units of light-meters per meter of time. Thus,

$$t^2 - x^2 = 0$$

A spacetime separation of zero we call a lightlike interval ("null geodesic"). Now consider a more complex situation: I place two assistants, one to my right, one to my left, at equal distances from me. Both are instructed to flash their lights when they see mine flash. [Draw the spacetime diagram on p. 176]

Now observe these events from a rocket that moves in the positive x direction through my lab. What does it see? Use the Lorentz transformation:

$$t'_R = -v_{rel}\gamma x_R + \gamma t_R = -v_{rel}\gamma x_R + \gamma t_R$$

Note that γ does not blow up: it's the signal that moves at $v = c$, not the rocket! For the flash that goes off to my left:

$$t'_L = -v_{rel}\gamma x_L + \gamma t_L = -v_{rel}\gamma(-x_L) + \gamma t_L$$

Thus, the flash to my right appears to occur before the flash to my left.

Likewise, a left-moving rocket sees the left flash go off before the right flash.

2 Light Cone: Partition in spacetime

With our knowledge of different kinds of intervals between events, let's look again at the spacetime diagram. Our goal is to examine how events are causally connected in spacetime. In other words, which events can affect each other? Which events lie in the past of a particle, thus having already affected it, and which events lie in its future, that it might affect?

Note that causal connection does not mean that the particle has to travel through that event, only that some kind of signal could be transmitted between the particle and that event. [DRAW diagram on p. 181, but supress second spatial dimension]

Let's put ourselves at $x = 0$, $t = 0$ in this diagram. What happens if we send light signals to observers at other locations, at $x > 0$? Those signals follow a path with $v = 1$ and the reception of those signals lies along the line $t = x$. Thus, the separation between transmission and reception of those signals has $t^2 - x^2 = 0$, so they are separated by lightlike intervals.

Same for observers out in the negative x direction. Reception events all occur along the line $t = -x$.

Suppose those same observers have the ability to send light signals to me. If I am now receiving a light signal from someone out at positive x, where does the event of him transmitting the light lie? Since light moves at $v = 1$, the transmission event lies somewhere down on the line $t = -x$. A signal from further away takes more time, and so that transmission event occurs further down on that line.

Likewise for observers out in the negative x direction. If I am now receiving light signals from them, the events occured somewhere down the line $t = x$.

What about signals that I send at slower than light-speed? This could include my getting in a rocket ship and delivering the message myself. The message moves at $v < 1$, so the reception of the event occurs somewhere between the lines $t = x$ and $t = -x$.

Likewise, if one of the other observers sent me a message in the past, and sent it at $v < 1$, the transmission event must lie at $t < 0$, somewhere between the lines $t = x$ and $t = -x$.

Thus, all of the events that currently affect me lie in my past somewhere along or in between those two lines. Can events outside those lines affect me? NO. The signal from those events to me would have to travel at faster than light speed!

What impact can I have on this spacetime? Since signals that I can send, whether they're light signals, bullets that I fire, or messages that I deliver by hand, must travel at $v < 1$, all of the events that I can cause in the future must lie at $t > 0$ in between those lines.

Can I affect points in the space time diagram that lie at $t > 0$ but below those lines? NO. That also would require faster than light travel.

So, at any location in spacetime, we can divide spacetime into several regions:

Active future Future events separated by timelike interval. In principle, I could travel there and bonk them on the head!

Future light cone Future events separated by lightlike interval. I can affect these events, but only with a light signal. Can't get there myself because I can't travel at $v = c$!

Passive past I am affected by events here, which are separated from me by timelike interval, so material objects can get to me from there.

Past light cone Past events separated from me by lightlike interval. Signals can only get to me from there at light speed.

Unreachable, unaffectable $t > 0$ It's too late for me to alter any of these events. But, at some point in MY future, I may be affected by these events.

Unreachable $t < 0$ These events can't affect me right now. But they could in the future. Obviously, I can't alter these affects.

3 More on Spacetime Diagrams

Remember, that the light cone defines the past and future relative for a particular point in spacetime. Even if I stay at the same location, at some future time the past light cone and future light cone will include a different set of events! (DRAW THIS.)

Why is it called a "light cone?" Well, we've been drawing spacetime considering only one dimension. If we add another spatial dimension, then the spacetime diagram and the light cones look like this: [DRAW diagram on p. 181]

In the 1d diagram, the light cone is defined by the lines with zero spacetime interval $t^2 - x^2 = 0$, thus $t = x$ and $t = -x$. In 2d, this is

$$t^2 - (x^2 + y^2) = 0$$

$$t = \sqrt{x^2 + y^2}$$

This defines a cone, hence the name. In 3D,

$$t = \sqrt{x^2 + y^2 + z^2}$$

which describes hypersurfaces in 4D spacetimes (I can't draw those!).