INTRODUCTION TO RELATIVITY
Winter 2018-2019
Prof. Michael S. Vogeley

Lecture Notes #6: Trekking Through Spacetime
Thursday, February 9, 2019

0 Preliminaries

Required Reading: *Spacetime Physics*, ch. 5

Homework: Midterm next Tuesday, February 12.

Concepts to review/learn: Following are the essential ideas of the next section of the course.

Spacetime diagram How to draw it. Depends on the reference frame.

Reference event Used to mark zero of time and space in all overlapping free-fall frames.

Invariant hyperbola Invariance of spacetime interval between the reference event and any other event, $t^2 - x^2$, describes a hyperbola in spacetime diagram. Transforming an event to another frame will land it somewhere along this hyperbola.

Worldline Connects events that occur at location of some object, e.g., a particle.

Proper time Elapsed time as measured by a clock that travels along the worldline with the object (particle). Proper time is also the length in spacetime of the worldline and the amount by which the observer ages.

Maximal aging Occurs for observer who travels on a straight line between two events. Along any path that curves, there will be less aging.

To deviate from the straight path, which is what a free particle would follow, you need to accelerate. This clarifies why the astronaut who travels out and back is the one who ages less.

Stretch factor Ratio between elapsed time between events as measured in a reference frame to the proper time along a worldline in which a particle moves at speed v in that frame.

1 Spacetime Diagrams

We used spacetime diagrams earlier when we talked about the "Trip to Canopus." Now let's consider them in more detail. Some Key Points/Questions:

- Spacetime map is specific to a particular free-falling frame.
- What are invariants in spacetime diagrams?
- What do distances between events mean in spacetime diagrams?

We want to make a map that shows where events take place in spacetime. Not just a space map, a spacetime map. Not just a calendar of events, but a record of where and when each event takes place.

What's a space map? Look at a highway map. It shows how to get from Philly to New York, or from Paris to Houston. You could draw your trip on this map with a marker. This would be a space map, showing the location of all the events on your trip. But that doesn't say WHEN you were at different points in space. You could keep a diary of your trip, recording the times and dates of your visits to different place. This would be a record of the times of these events.

What about a spacetime map of your trip? Can you combine the locations and times of all the events on your trip into a single description of your trip? Sure, with some simplifying assumptions. Spacetime is inherently four-dimensional: 3 dimensions of space plus one of time. I don't know how to draw maps in 4 dimensions and certainly not how to draw the 2 dimensional projection onto the chalkboard of a 4 dimensional space. Let's keep it simple and deal with 1 dimension of space, so that spacetime is only 2 dimensional.

2 Super Bowl Sunday

Let's DRAW a simple spacetime diagram in my living room free-fall frame. Space is on the horizontal axis. Time is on the vertical axis. We set the zero of the coordinate system at some specified time and place (an event). Now let's mark some events:

0 reference event at $t = 0, x = 0$

A $x > 0, t = 0$

B $x = 0, t > 0$

C $x > 0, t > 0$

D $x < 0, t > 0$

What are these events?

0 is the reference event: I hit the remote control to turn on the Super Bowl pregame show. Spacetime now revolves around ME.

A is an event that happens at the same time as the reference event, but at a different place. My brother John, sitting at other end of couch, opens his first beer at the same time as I hit the remote control.

B is me at the same place, just later in time. Just sit around at the origin and wait for it. I hit the remote control to turn up the volume.

C occurs at a different place, later in time when John opens another beer.

D an event at negative x. Nothing special about positive and negative x. The door bell rings, announcing arrival of pizza.

This spacetime diagram only works for a particular reference frame. If these are all events with times and locations as measured in my laboratory, an observer flying by in a rocket would record different times and places, thus we would draw a very different spacetime diagram, right? We'll come back to this in a moment.

Imagine that the doorbell is actually a flashing light, sending me a signal announcing the arrival of pizza. The signal travels from D to E at light speed, its worldline at a 45 degree angle. Then I walk from E to F, along a line more vertical than 45 degrees, then back to the couch (Aaaaah).

This is a spacetime diagram, not a space diagram. At any given moment of time, we can draw a slice through this picture to make a spatial map of where all the people and objects are at that instant t. Likewise, we can make a calendar, a time-sequence of events, for any location by drawing a slice at fixed x.

Notice how causality works here. There is no way for me to get the pizza before the TV gets turned on. Why? The pizza would have to travel backwards in time! (How many calories in a tachyonic pizza?)

3 Same events in different free-float frames

Now let's examine how the same events appear in the spacetime diagrams of different free-fall frames. We'll use the example of light bouncing between two mirrors inside a rocket ship. We used this example before to derive time dilation.

[see diagrams from book, pp. 140-141]

Let's DRAW the rocket ship with light, mirror, detector. Light travels up 3 meters, hits mirror, back down 3 meters. Events E = emission of light, R = reception of light.

What does observer in lab see? (DRAW laboratory frame.) Light travels up 3 meters, over 4 meters, down 3 meters, over 4 meters.

What does observer in rocket travelling faster than first rocket see? To him, the light moves back through his frame. DRAW super-rocket frame. Light starts near front of ship, travels up 3 meters, back 10 m, off mirror, down 3 m, back 10 m.

Now look at spacetime diagrams of each. All three frames overlap and we set the zero of space and time for all three at the moment that the light is emitted.

Rocket frame: Events E, R occur at same place, different time, because apparatus just sits there in rocket frame.

$$x = 0, t = 6\,\mathrm{m}$$

Lab frame: Event E at origin, event R at positive x, larger t due to time dilation. Use spacetime interval:

$$\begin{aligned} t'^2 &- x'^2 = t^2 - x^2 \\ t'^2 &- (8\,\mathrm{m})^2 = (6\,\mathrm{m})^2 - (0)^2 \\ t'^2 &= 36 + 64 = 100, t' = 10\,\mathrm{m} \end{aligned}$$

Thus

$$x' = 8\,\mathrm{m}, t' = 10\,\mathrm{m}$$

Super-rocket frame: Event E at origin, event R at negative x (because super-rocket was overtaking the rocket while light was moving) and even larger t, since difference between rocket and super-rocket velocity is larger than difference between rocket and lab.

$$\begin{aligned} t''^2 &- x''^2 = t^2 - x^2 \\ t''^2 &- (-20\,\mathrm{m})^2 = (6\,\mathrm{m})^2 - (0)^2 \\ t''^2 &= 400 + 36 = 436, t'' = 20.88\,\mathrm{m} \end{aligned}$$

Thus

$$x'' = -20\,\mathrm{m}, t'' = 20.88\,\mathrm{m}$$

4 The Invariant Hyperbola

First we'll look at the case of events with *timelike* separations. These are pairs of events than an object could travel through.

Invariance of the spacetime interval between events means that

$$t^2 - x^2 = t'^2 - x'^2 = t''^2 - x''^2$$

That describes a hyperbola in the spacetime diagram. We call this the "invariant hyperbola." Given the location of an event, we know that transforming that event to any other free-falling frame will find it somewhere on that hyperbola.

Note that the hyperbola lies between lines at 45 degrees, asymptotically approaching those lines. Transforming from one free-falling frame to another causes events to slide along such a hyperbola. But events never move below those lines.

Note that distance between events in the spacetime diagram DOES vary. That's because the distance as it appears in the diagram is

$$d^2 = t^2 + x^2$$

But that's not how we measure distances in spacetime, using

$$d^2 = t^2 - x^2$$

5 Proper Length and Proper Time

Proper length = length measured of object at rest in a frame, between two events at same time, along a horizontal line. The ends of an object at rest follow vertical lines in the spacetime diagram.

Proper time = time measured between two events that occur at same place, along a vertical line.

What happens when we transform events to another frame? Look at proper time first: transformation of events E, R in example above is transformation of proper time in rocket frame to lab frame and super-rocket frame. Proper time is the minimum time. Transforming to another frame makes the time between events longer, hence the name "time dilation." In the other frame, the second event occurs at a different time AND place.

What about proper length? We'll discuss this in detail next week, but let's peek ahead now: Note that the interval here is

$$t^2 - x^2 = (0)^2 - x^2$$

This means that any other frame has

$$t'^2 - x'^2 < 0$$

(This is obviously a *spacelike* interval.) So the transformed event must lie along a hyperbola bounded by another pair of 45 degree lines. Draw another hyperbola, rotated by -90 degrees. Wait! Didn't this make the distance interval longer in the new frame? Isn't the SR effect called "length contraction"? Yes, it is. But remember how we measure lengths. We simultaneous mark both ends of an object or simultaneously note the location of two events. Of the two events in question here, one of them is at the origin, which by construction has the same coordinates in all frames. Thus, transforming the events that define a proper length in one frame does not yield events that are simultaneous in another frame.