INTRODUCTION TO RELATIVITY
Winter 2018-2019
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Lecture Notes #5: Trip to Canopus
Tuesday, February 5, 2019

0 Preliminaries

Required Reading: *Spacetime Physics*, ch. 4

Homework: HW4 is due Thursday, February 7:
Just 3 problems from chapter L: 6 (a only), 7, 8

1 Proposal: A Trip to Canopus

Good news! NASA wants to send a manned mission to the unusual star Canopus, to make astronomical observations as close to its surface as we can get, and return home.

Bad news (maybe): you have been selected to make the trip! This trip to Canopus will not involve actually landing on the star. Look, but don't touch!

The problem is that Canopus lies 99 light years away. Even at light speed, it would take 99 years to get there and 99 to get back, right? We would need to take lots of people and return 6 or 7 generations (roughly, 6 times 30 or so) later. Pack the wife and kids, pigs, chickens, some good books... Not likely that we would want to do this. Or maybe there is a way...

2 Measuring speed along the way: Faster than light?

To decide if we'd like to actually make the journey, let's first imagine what the trip would be like, yet another "Gedankenexperiment."

Imagine that along the way from Earth to Canopus are strewn some beacons that show the time in the Earth reference frame, one at every light-year checkpoint. These beacons are synchronized in the Earth frame just like the camera-clocks in one of our rod and clock lattices. How did they get there? Don't worry, this is just a thought experiment, not a real one! Now let's imagine that we take the trip on a rather fast spacecraft. On board the ship we carry own our very accurate atomic clock to time the journey.

We leave Earth on a momentous occasion, at noon on July 4, 2000, with a marching band playing for us and a big send-off parade. As we travel along toward Canopus, we peer out the window looking for the beacons in space that mark the way. When we see one, we compare the time and date on our clock to the time and date on the beacon.

(DRAW Earth, Canopus, rocket, numbered beacons along the way)

At beacon #8, 8 light years from Earth (as measured in the Earth frame), we see that the beacon time is 12:00 PM, July 4, 2010. We look down at our clock, which reads 12:00 PM, July 4, 2006. Hmmm. What's going on here? Have we travelled 8 light years in 6 years of time (as measured on our clocks), for a speed $v = 8/6$? Or have we travelled 8 light years in 10 years (as measured by the beacon clocks), for a speed $v = 8/10$?

What's the correct way to measure velocity? We have to measure both distance and time in the same reference frame! In the rocket frame, the time of our trip so far is as measured by the clock on board the ship, $\Delta t' = 6$, and we have travelled a distance $\Delta x' = 0$ – because we're just sitting inside the rocket. Thus our speed in the rocket frame is $v' = 0$, right?

In the frame of Earth, we have travelled a distance $\Delta x = 8$ in a time $\Delta t = 10$. Thus an Earth-frame observer says that we're moving at speed $v = 8/10 = 0.8$ times the speed of light. This is also the speed at which Earth appears to move from our perspective; v_{rel} must be the same as measured by both frames.

You have to be very careful to keep track of which measurements belong in which frame!

3 The Flight Contract - Carefully State Your Terms

We have the notion that some SR effect involving time dilation might help us get to Canopus and back in less than twice 99 years. In our previous thought experiment, travelling at $v = 0.8$, we covered 8 light-years of Earth-measured distance in 6 years of rocket-clock time, but 10 years of Earth time, thus we anticipate that the trip would take $99 \times 6/10 = 74$ years in each direction ($1/\gamma = 6/10$ is ratio of times from above). Less than 99, but still too long for our taste. Going faster might decrease the length of the trip as measured by the rocket clock and, therefore, as measured by our body clocks. But a faster ship will cost NASA more money!

Let's make the length of time for the journey a demand to NASA - a term of our contract with them. But we have to state it very carefully. Let's say that we don't want to spend more than 20 years to get there, as measured by the clock on board the rocket ($2 \times 20 = 40$

years - come home to retire). The DISTANCE of the trip will be specified as measured in the Earth frame, but the TIME elapsed for the trip, which is what we care about, must be short enough in the rocket frame for us to return alive!

After playing with the numbers a bit, we propose that NASA build a ship that will travel at 99/101 times the speed of light. Preposterous, they say. Well, that's what we need, we reply. Let's justify this request.

Consider the spacetime interval in the Earth frame, evaluating two events: launch from Earth and turnaround at Canopus. At $v = 99/101$, it will take

$$t = 99\,\mathrm{ly}/(99/101\ \mathrm{ly}/\,\mathrm{y}) = 101\,\mathrm{y}$$

to travel in each direction. The spacetime interval between launch and turnaround, as measured in the Earth frame, is

$$t^2 - x^2 = (101)^2 - (99)^2 = 10201 - 9801 = 400 = (20)^2\,\mathrm{y}^2$$

Thus the spacetime interval is 20 years.

The invariance of the spacetime interval says $t'^2 - x'^2 = t^2 - x^2$, where the primes are the rocket-frame time and distance. In the rocket frame, we don't move at all, thus $t'^2 - (0)^2 = (20)^2$, thus the time elapsed in the rocket frame is simply 20 years.

Thus, the time dilation factor is $101/20 = 5.05$. Note that this is the Lorentz factor for $v = 99/101$, $\gamma = 1/\sqrt{1 - (99/101)^2} = 5.05$.

How long does the trip take in the Earth frame? The whole journey will take 202 years in the Earth frame. Yikes! NASA doesn't like this, but we explain that this is tough luck. Even if we travel at light speed, the trip could be no shorter than 198 years as measured on Earth. The fact that the trip lasts a much shorter time as observed on the rocket doesn't change the situation on Earth very much.

We explain that the trip could be even shorter for us: Suppose that the ship could travel at $1 - 10^{-6}$ of lightspeed. Then the trip would last only 0.28 years, or about 3.5 months! But it would still take about 200 years in the Earth frame.

The lesson here is that, with a fast enough ship, we can travel to any place that we want in an arbitrarily short period of time. But, as we'll discuss, this comes at a price – we don't return to the same TIME. Time travel is quite possible, but only time travel to the future.

NASA is not happy about the 202 years before they get their data, but they're encouraged that we're willing to take the trip, provided that it only ages us by 40 years.

4 Challenges to our Plan

Our plan is submitted to a panel of experts and politicians (these groups are, of course, mutually exclusive) for review.

4.1 Two Inertial Frames?

Question from Jim Fastlane:
"How is it that you will age only 40 years while we here on Earth will age 202 years? That's ridiculous! You say that this happens because of Special Relativity, because you will fly at almost light speed. But once you're flying along toward Canopus, we could just as easily say that you're standing still and Earth is moving – they're both perfectly good inertial frames. Thus, your proposal is nonsense."

Our response:
It's the fact that we have to turn around and come back that makes the rocket flight different from the Earth frame.

Fastlane:
"What? Now I really know that you're crazy. You age less because you're out driving around?"

Well, actually that IS the answer! Look at the spacetime interval: $d^2 = t^2 - x^2$. For the same total spacetime interval, a longer path travelled implies a shorter amount of time elapsed (in the frame of the traveler, that is). So, the surest way to get old at the maximum rate is to sit around and do nothing. Makes you want to hit the road, doesn't it? It almost doesn't seem fair – the faster you go, travelling around the Universe, seeing lots of things, the slower you age!

Look at the spacetime diagram of the journey in the Earth frame (DRAW it!). Consider all the possible paths one can take between two events: launch of the rocket and return of the rocket. The spacetime interval between the two events is the same for all, but the time elapsed for the traveler is shortest when the spatial distance traveled is longest! The longest possible distance traveled is when the traveler moves about at nearly the speed of light.

Remember to use the invariant spacetime interval:

$$d^2 = t_{Earth}^2 - l_{Earth}^2 = t_{Rocket}^2 - l_{Rocket}^2$$

The distance between events is zero in the Rocket frame (they both happen at the rocket), $l_{Rocket} = 0$, and the Earth frame distance between the events is simply $l_{Earth} = v_{rel} t_{Earth}$, thus the elapsed times in the Earth and Rocket frames are related by

$$t_{Rocket} = t_{Earth}\sqrt{1 - v_{rel}^2}$$

4.2 Lorentz Contraction

One of the other panel members chimes in, this time Dr. Joanne Short:
"Wait a minute, you say that only 20 years will elapse on your outward journey. But you STILL know that the distance is 99 light years. How can you travel 99 light years in 20 years?"

Any suggestions, class?

Of course, Dr. Short is getting very confused by mixing up measurements from different reference frames. She's comparing the 20 years of time in the rocket frame to 99 light-years of distance measured in the Earth frame. But that's NOT the distance for the trip as measured in the rocket frame! From the rocket frame, we'll measure a different Earth-Canopus distance, which is shorter due to the SR length contraction effect.

There's no problem, because the distance from Earth to Canopus appears shorter in the rocket frame by the same factor that the time appears shorter. The ratio of rocket time/Earth time for the trip is 20/101. The length contraction has the same ratio, so from the rocket we'll measure the distance from Earth to Canopus as $99 \times 20/101 = 19.6$ light-years. We can get there in only 20 years because, to a rocket moving at 99/101 of light speed, the journey only looks 19.6 light-years in length!

4.3 Travel into the Future – but Not Back!

It occurs to one of the panel members that we'll be travelling forward in time, returning to Earth 202 years later, about 6 human generations later. Of course, we knew that and accepted that we'll never see our immediate families again – maybe we should bring them on board?

Let's draw a spacetime diagram to show what happens. Our world line is at nearly 45 deg angle going out and coming back. We return far into the future. There's no way to get BACK in time, only to go forward. Note how spacetime interval is measured in this space: $d^2 = t^2 - x^2$, NOT $t^2 + x^2$

Review: $v_{rel} = 99/101$, $\gamma = 101/20$, $t_{Earth} = 101\,\mathrm{y}$, $L_{Earth} = 99\,\mathrm{ly}$, $t_{Rocket} = 20\,\mathrm{y}$, $L_{Rocket} = 19.6\,\mathrm{ly}$.

4.4 Relativity of Simultaneity - Jumping from Frame to Frame

Are we done justifying our plan to NASA? No, our critic Jim Fastlane has one last withering criticism.

Fastlane:
"Let's look again at comparing time in rocket and Earth frames. While it's flying along at constant velocity your rocket frame is an inertial frame just like the Earth frame. You're sitting at rest in the rocket. Imagine a set of synchronized clocks in the rocket frame, stretching out as long at the Earth-Canopus distance. Since the Earth moves relative to the rocket-frame clocks, a clock on Earth should appear to run slower than the rocket-frame clocks. So that trip takes less time on the Earth clocks than it does on your rocket clock. The same thing happens on the way back: the time measured by the Earth clock, which appears from the rocket frame to be moving, will be less than as measured by the rocket clock. So, tell me again how you do this in only 20 years?"

Uh-oh. It would seem that he's got us here. Or does he? Wait a minute! Our trip to Canopus and back involves *more than one inertial frame*. We fly out at constant velocity in one direction, but we return at constant velocity in the opposite direction. Those are TWO separate inertial frames. At Canopus, the process of braking and turning around effectively jumps us from the outgoing inertial frame to the incoming inertial frame.

Each rocket-frame, incoming and outgoing, has a large set of synchronized clocks attached to it. From the rocket frame, note the time on a clock at Earth while we travel to Canopus and back. We have to be very careful to note WHERE the clock is and in what frame we are making the observation.

Let's analyze our trip again. We agree that, as measured on board the rocket, the trip to Canopus takes only 20 years if we travel at 99/101 lightspeed. What does the Earth clock read? It will appear to run slower than the rocket-frame clocks, and the relative velocity between the rocket and beacons is 99/101, so FROM THE ROCKET FRAME, the Earth clock appears to run slow by the same factor that the rocket clock appears to runs slower as observed by the Earth frame. Thus, as observed by the rocket frame, the Earth clock records a time $20 \times (20/101) = 3.96$ years. Note that this observation appears very different from our Gedankenexperiment, in which we looked out the window and looked at the time on beacon-clocks in the Earth frame. Why? Because although the Earth clock and all the beacon clocks appear synchronized in the Earth frame, they do NOT appear synchronized from the rocket frame.

Remember the key to this: Rocket clock appears to run slow, as observed by Earth-frame. Likewise, the Earth clock appears to run slow as observed by the rocket-frame. From the *outbound* rocket frame, the Earth clock hits 3.96 years just as we reach Canopus. As measured on Earth, that same event occurs after 101 years. The discrepancy is due to the relativity of simultaneity! The rocket and Earth frames DO NOT record the arrival of the rocket at the same time.

Now, what happens when we turn the ship around? We jump to the incoming inertial frame. During the return trip, again our rocket clock records 20 years of time, but from the rocket frame, the Earth clock ticks off 3.96 years of time while we reach Earth again.

But observers on Earth tell us that our roundtrip took 202 years, right? That means that, at the moment after we turned around, Earth clocks must have read $202 - 3.96 = 198.04$ years. But didn't it take only 3.96 years on the Earth clock for us to get there?

Remember, the 3.96 years of outward time is the time that the outgoing rocket observer sees on the Earth clock. 198.04 years is what the incoming rocket observer sees on the Earth clock. They are NOT in the same inertial frame! The jump from outgoing to incoming inertial frame causes a difference in what the rocket observer sees. This is because the outgoing and incoming inertial rocket frames have different notions of simultaneity!

Simultaneity is important: In the outbound rocket frame, we say that we arrived at Canopus and checked the Earth clock at the same time. But those events are not simultaneous in the Earth frame. Those events are also not simultaneous in the incoming rocket frame.

The fact that we had to jump from on inertial frame to another makes our rocket trip special, thus breaking the symmetry between the paradox of "Earth sees rocket moving but rocket sees Earth moving."

Each of the three inertial frames, Earth, outgoing rocket, and incoming rocket are equally valid. It's our changing from one frame to another that jars the simultaneity of measurements and distinguishes the rocket trip from simply sitting on Earth.

The jump from frame to frame does require an acceleration. During this time, we're no longer in an inertial frame, thus SR does not necessarily apply. Does this mean that we need General Relativity to explain the paradox? No, not really. We just need to change inertial frames very slowly. Approximate the "jump" by a set of small steps between several inertial frames.