PHYSICS 233: INTRODUCTION TO RELATIVITY
Winter 2018-2019
Prof. Michael S. Vogeley

Lecture Notes #10: Gravity: Curved Spacetime in Action

0 Preliminaries

Required Reading: *Spacetime Physics*, ch. 9

Topics for today

- Gravity is not a force, it's the curvature of spacetime
- Equivalence principle
- Gravitational time dilation and redshift
- The Twin Paradox - again
- Black holes - no escape and the no hair theorem
- Gravity waves

1 Gravity: Newton vs. Einstein

According to Newton:
Space is Euclidean. Set up a perfect 3D coordinate system that extends as far as necessary. Curved paths in this system reveal accelerations, which are caused by forces. Recall Newton's First Law: bodies at rest remain at rest, bodies in motion remain in motion, unless acted upon by an outside force. The Newtonian picture allows "action at a distance." Because there is no accounting for the finite speed of transmission of information, a body in one location acts at a distance to cause a force on another body *instantaneously*.

According to Einstein:
Space is Riemannian. Curved, that is. Riemann was the mathematician who developed much of the mathematical tools used by Einstein to derived the GR equations. It makes no sense to describe physics with a globally Euclidean background. How does one define the straight lines in that Euclidean grid? What travels in a straight line? Maybe light? Just try it. Shine a light between two distant points. But even light changes its path when it passes near a

massive object. Or does it? What does "change its path" mean? Change relative to what? The light is in its own freely-falling frame - it feels no "acceleration."

Likewise, sit in your rocket ship and travel along a "straight line" through space. If you don't fire your rockets, the ship carries along and you feel no acceleration. Travel out around the moon and back. You don't feel any "acceleration," no "force" as you travel through the solar system.

Einstein says, why talk about gravity as a force that causes acceleration, when nothing of the kind occurs locally? (See discussion of the "Strong Equivalence Principle below.) This is one of the motivating ideas in relativity: "Physics is simple only when when analyzed locally." And locally (in a small enough patch of space), spacetime is Lorentzian – its the spacetime that we've used in Special Relativity, with distance interval $(d\tau)^2 = (dt)^2 - [(dx)^2 + (dy)^2 + (dz)^2]$

Remember, "locally" means local in both space and time. At the beginning of the course, we studied freely-falling frames. For example, we looked at a spaceship that travelled around the Moon and back. Over long periods of time, the traveller in the ship could detect the effect of tidal forces within the ship. Remember our discussion of tennis balls inside the ship. But, for small enough volumes and short enough times, the inside of the ship is a freely-falling frame. Spacetime inside the ship is Lorentzian, and Special Relativity works, for small enough volumes of spacetime.

General Relativity links up all of those little freely-falling frames. Locally, spacetime is simple. But globally, it is curved – the ship goes around the Moon and back!

2 Travelling in curved space

The difference between local and global curvature may be understood by considering motion on the surface of the Earth. Locally, the surface looks quite flat, but most of us recognize that the Earth is round!

Consider two cars heading North on the Earth. Suppose that they both start on the equator, separated by 1000km in the East-West direction. Both drive straight North for 1000km. They are closer when they stop than when they began. But neither felt any acceleration to the left or right. Looking down from far above, assuming that the Earth's surface is flat, we would conclude that there was some force pulling them together. But there's no such force, just the curvature of the Earth. Each car followed a line that always appeared locally straight and neither car ever felt any force.

Now, keep this car example in mind as we consider a freely-falling frame near the Earth. Our spaceship returns to Earth. As it nears Earth, but before entering the atmosphere, we let go of two balls at opposite sides of the ship, both equal distances from the Earth's center.

As time passes, they get closer. We infer that some force is pulling them together.

DRAW the spacetime diagram of the following: [figure 9-5, page 283] The path of each ball is curved in spacetime. We can break up this curved path into many "straight" segments. The inside of the ship is a good freely-falling or inertial frame for as long as we can't detect departure of ball's motion from constant velocity. Each segment is straight and the ball moves at constant velocity. But the full (global) path is curved. Does the curve of the ball's path from straight lines of our spacetime diagram mean that a force is acting on the balls? Locally, there doesn't appear to be anything pushing or pulling on the balls, yet they move. The general relativistic interpretation of this phenomenon is that there is NO force. The balls are indeed falling freely through space, following the curve of space that the Earth causes. The motion of a single ball would not reveal this curvature. Only the apparent relative motion of two balls following slightly different paths reveals that space is curved.

What is it that causes spacetime to be curved? Momenergy! The fundamental equation of GR is Einstein's equation:

$$G^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}$$

On the left is the Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

which encodes information about the curvature of space. In particular note the metric tensor $g^{\mu\nu}$ that describes the coefficients in front of the spacetime displacement

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where dx^μ is the spacetime displacement 4-vector. In the spacetime of Special Relativity,

$$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

(that's shorthand for a matrix whose only non-zero elements are on the diagonal, as listed) so the spacetime displacement is simply

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

On the right side of Einstein's equation is the energy-momentum tensor. This tensor includes contributions from the rest energy of matter, momentum, etc. It's the generalization of the momenergy 4-vector. So, the right side – matter and energy – tells space how to curve. In turn, the curvature of space tells matter and energy how to move!

The zero-zero component of the energy-momentum tensor (also called the stress-energy tensor) is just the energy density,

$$T^{00} = \rho$$

where ρ includes all forms of mass-energy. For a perfect fluid, the whole tensor looks like

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

3 Equivalence principles

In Newtonian terms, the acceleration of a body in a gravitational field is $m_i\mathbf{a} = m_G\mathbf{g}$. Experiment verifies that the inertial and gravitational masses are identical to better than one part in 10^{11}. This led Einstein to suggest that gravitational and inertial forces are the same thing! He introduced the "weak equivalence principle." This says that a freely-falling observer will experience no gravitational effects – with the important exception of tidal forces in non-uniform fields. We saw these tidal forces revealed in our trip around the Moon. The spacetime of the freely-falling observer is exactly that of SR.

The "strong equivalence principle" takes this one step further. Not only is spacetime in a freely-falling frame the same as in SR, but also all the laws of physics look the same in a freely-falling frame as they do in the *absence* of gravity. Conversely, an accelerating reference frame is identical to a frame that is at rest in a uniform gravitational field.

4 Gravitational time dilation

To illustrate an important implication of the equivalence principle, let's look again at physics inside a rocket ship. But, this time, instead of setting the rocket ship in motion at constant velocity and letting it go, let's fire the rockets so that the rocket accelerates with $a = g$. Thus, the rate of acceleration is the same as that felt by a particle falling at the surface of the Earth.

Inside the rocket, place a clock on the ceiling, a distance h from the floor. Sit the traveller on the floor with his own clock. The ceiling clock works by flashing once a second. To make things easy, let's suppose that it emits just one photon each flash. What happens while the photon travels from the clock to us, sitting on the floor? It takes time $t = h/c$ for the photon to reach the floor. In this time, the rocket speeds up by an amount $v = gt = gh/c$.

Again, at the moment the photon was released, you could think about the rocket as sitting in a freely-falling frame that moved with the same velocity as it had at that moment. By the time the photon reaches the floor, the rocket has velocity v in that frame. The floor moves while each photon is in flight, so the photon has to travel a shorter distance, $\delta x = vt = (gh/c)(h/c)$, and so take less time to travel, $t' = (h - \delta x)/c$.

From the standpoint of the observer on the floor, this means that the photons arrive sooner than expected by a fractional amount $\delta t/t = gh/c^2$. The observer on the floor must conclude that the clock on the ceiling is running fast – it is spitting out photons at a rate faster than once per second as measured by the clock on the floor. Imagine that you're the observer: you climb up a ladder to the ceiling and compare your watch with the clock – they run at the same rate. You use your meter stick to check that the spacing between the rungs does add up to a height h for the inside of the rocket. You climb down the ladder a bit and notice that the ceiling clock runs fast. You climb back up to find that, indeed, more time has elapsed on the ceiling clock than on your watch. This was not an illusion; time really does run faster higher up in the rocket.

5 Gravitational Time Dilation (continued)

Recall our discussion of how clocks run faster higher up in an accelerating rocket. The equivalence principle tells us that the accelerating rocket is identical to a rocket sitting on the surface of the Earth. Thus, the difference in gravitational potential between the floor and roof of the rocket $\Delta\phi = gh$ causes time dilation $\Delta t/t = \Delta\phi/c^2$. Another way to write this, accurate for weak gravitational fields like near the Earth, is that

$$\frac{dt_1}{dt_2} \approx 1 + \frac{\phi_1 - \phi_2}{c^2}$$

where ϕ is the gravitational potential, $\phi = -GM/r$. This is more negative closer to the Earth, so time runs slower on the surface than far out in space.

Therefore, clocks run FASTER in orbit around the Earth than on the surface! Remember that the SR effect is that clocks in a fast-moving airplane or satellite should run SLOWER. So these effects have the opposite sign, but don't cancel.

Another way to think about time dilation is by looking at the effect of the curvature of spacetime. In the spacetime around a spherical mass (using $c = 1$), the spacetime interval is not simply $(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$. The curved spacetime looks like (neglecting the angular coordinates and spatial separation only in the radial direction)

$$(d\tau)^2 = \left(1 - \frac{2GM}{r}\right)(dt)^2 - \left(1 - \frac{2GM}{r}\right)^{-1}(dr)^2$$

where r is radial distance from the center of the mass. Again note that we set $c = 1$ (else the terms look like $2GM/c^2r$). Pick two events that are separated by proper time $d\tau$. Here I'll put the c^2 factors back in. If they occur at the same r (like two ticks on a clock), then $dr = 0$ and

$$d\tau = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$$

or

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \approx 1 + \frac{GM}{c^2 r}$$

Using

$$\frac{1}{\sqrt{1-\epsilon}} \approx 1 + \frac{\epsilon}{2}$$

Clearly, $(dt/d\tau) \to \infty$ at $r = 2GM/c^2$ and $dt/d\tau \to 1$ as $r \to \infty$. Since $dt > d\tau$, this means that a clock at distance r from the center of the mass runs slow by the factor on the right. At infinite distance from the star, $dt = d\tau$. This expression is the exact version of the weak limit. (The latter was derived by noting that $(1-x)^{-1/2} \approx 1 + x/2$ when $x \ll 1$.)

In other words, an observer who sits deep within a strong gravitational field see distant clocks run fast, while a distant observer, at $r = \infty$, see clocks near the star run slow. Using the approximation in the equation above for $dt/d\tau$ and noting that the proper time interval is invariant, the ratio of clock times at two distances from the star is

$$\frac{t_1}{t_2} \approx \frac{1 + \frac{GM}{c^2 r_1}}{1 + \frac{GM}{c^2 r_2}}$$

Using $1/(1+x) \approx 1 - x$ for $x << 1$,

$$\frac{t_1}{t_2} \approx (1 + \frac{GM}{c^2 r_1})(1 - \frac{GM}{c^2 r_2}) \approx 1 + \frac{\phi_1 - \phi_2}{c^2}$$

where the last approximation follows by keeping only terms of order $1/c^2$ or lower (surely the $1/c^4$ term is tiny!) and plugging in the definition of the gravitational potential. Now we see clearly that the gravitational time dilation is related to the change in the gravitational potential, $\Delta t/t = \Delta\phi/c^2$.

Note that the gravitation time dilation also implies gravitational redshift of photons that are emitted from a star into outer space. Since clocks run slower on the surface of the star than at infinity, there is more time in between crests of the EM waves, thus longer wavelength.

The gravitational redshift can be seen by comparing the frequencies of light emitted near the star and observed far from it. Clocks near the star run slower by the factor

$$\frac{t(r)}{t(r=\infty)} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$

Frequency is cycles/unit time, so the light from the star appears at large distance to have a lower frequency,

$$\frac{\nu_{\text{observed}}(r=\infty)}{\nu_{\text{emitted}}(r)} = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2}$$

The speed of light is constant and $c = \nu\lambda$, therefore the wavelength is longer by the factor

$$\frac{\lambda_{\text{observed}}(r=\infty)}{\lambda_{\text{emitted}}(r)} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$

Conversely, an observer deep in a gravitational potential (e.g., very near the star) sees light from distant sources *blueshifted*.

6 The twin paradox resolved using GR

We can use the gravitational time dilation to resolve the twin paradox! Twin A remains on Earth. Twin B travels to Canopus, at distance d, and back at speed v. The turnaround at Canopus is done at acceleration g. To make the math easier, let's assume that v/c is not very large, so that $\gamma^{-1} = \sqrt{1 - v^2} \approx 1 - v^2/2$.

On the outward trip: According to B, A's clock runs slow $t_A = t_B/\gamma \approx (d/v)(1 - v^2/2)$

On the return trip: According to B, A's clock runs slow $t_A = t_B/\gamma \approx (d/v)(1 - v^2/2)$

Neglecting the turnaround, A's clock runs slow by a total of $(d/v)(v^2) = vd$, thus $t_A - t_B = -vd$.

What happens at turnaround? To turn around, B's ship must fire its rockets to accelerate back towards A. Suppose it accelerates at $a = g$. To B, this must appear the same as being in a gravitational field with strength g, with A above it.

To turn around, velocity must change by $2v$. Recall $\Delta v = a\Delta t$ for constant acceleration. Thus, at constant acceleration g, it takes time $t_{turn} = 2v/g$. The Strong Equivalence Principle says that $\Delta\phi = gd$ (in units where $c = 1$ and recalling that $g = GM/r^2$). During this time, B observes A's clock to run fast by a fractional amount $\Delta t/t = gd$ (just as before, the difference in potential between A and B). Thus, the extra time elapsed on A's clock is $\Delta t = (\Delta t/t)t_{turn} = (gd)(2v/g) = 2vd$. In other words, during the acceleration B's frame observes that clocks at A run very quickly, as the line of simultaneity changes in the spacetime diagram.

Now add up the times: $t_A - t_B = -vd + 2vd \approx (\gamma - 1)t_B$. Thus, $t_B = t_A/\gamma$.

7 Compact objects

We mentioned that observations of neutron stars revealed gravitational radiation carrying off energy. What are neutron stars?

A brief primer on stellar evolution:
A normal star like the Sun will use up all its fuel, which means that it fuses H to He, then begins fusing He. When the core is all C and O, fusion stops and the Sun is dead as a source of energy. With insufficient pressure to support its weight, it collapses to a white dwarf. In a WD, only the quantum-mechanical repulsion of electrons keeps it from further collapsing.
Characteristics of a WD:
Radius = 5000km

Density = 10^9kg m^{-3} (note that water has density = 10^3 kg m^{-3})
Max. mass = 1.4 solar masses

When an even larger star burns out, the collapse of the outer regions causes huge pressure when it falls in, squeezing electrons and protons into neutrons. Such an object, a neutron star, is supported only by the quantum-mechanical repulsion of neutrons, which can be packed even more closely than electrons. Characteristics of a NS:
Radius = 10km
Density = 10^{17}kg m^{-3}
Max. mass = 1.4 solar masses

If the dying star is extremely massive, it fuses all the way to Iron in its core. When such a star collapses, the density in the center can get so high that a black hole is formed.

8 Black holes

Why is it called a black hole? Because even light cannot escape! What is escape velocity? For any spherical mass, we compute the escape velocity by equating kinetic and potential energy at surface, $K = V$, or $mv^2/2 = GMm/r$. Thus, the escape velocity is $v = \sqrt{2GM/r}$. If we make M large enough and/or r small enough, then we can make v_{escape} very large. At maximum velocity $v_{escape} = c$. If $v_{escape} > c = 1$, then nothing, not even light, can escape.

For a black hole, all of the matter collapses into a singularity, an infinitesimal point in spacetime. It has no "surface" that you could sit on. But it does have a boundary, so to speak. This is the distance from the singularity at which the escape velocity is the speed of light. Rearranging the equations above, we plug in c for the velocity and find that the radius is $R_S = 2GM$ ($R_S = 2GM/c^2$ in conventional units). This distance is called the Schwarzchild radius. Go closer to a black hole than this, and you're a goner!

What happens when matter falls into a black hole? It gets accelerated to very high velocity. Typically, the swarm of matter trying to fall in has some angular momentum and it forms a disk as it swirls around the black hole. The gravitational potential energy of the infalling matter is converted to kinetic energy in this disk, which reaches very large temperature.

The brightest objects in the universe, called quasars, are thought to be powered by the matter falling into supermassive black holes.

Now consider the behavior of the gravitational time dilation and the gravitational redshift as $r \to 2GM$: both tend to infinity. If you watch your friend fall towards the event horizon (Schwarzschild radius) of the black hole, he will appear to move slower and slower and you'll never actually see him fall in. Also, light from him will be increasingly redshifted until you cannot observe the photons (unless you have eyes that can see radio waves).

9 Gravity waves

Mass and energy cause curvature of spacetime. Likewise, that curvature tells mass how to move. But what happens when the mass then moves? The curvature of spacetime must change in response to the new location of the mass. But does spacetime change instantaneously? No. There is a time delay between moving the mass and changing the spacetime curvature. Ordinarily, this is a small effect. But for large masses and velocities, this becomes important.

Consider two masses, A and B. If we move A, when does B "feel" the difference in spacetime curvature caused by A's new position? The information that A has moved travels at the speed of light to B, no faster.

Consider what happens in Newtonian gravity: $F = -Gm_Am_B/r_{AB}^2$. This is what we call "action at a distance." There is no accounting in this equation for motion of A or B. The force depends only on the distance. This is wrong because it implies that information about A and B's position travels at superluminal speed.

Let's imagine a mythical story: Atlas, who ordinarily holds up the Earth, plays with two of Jupiter's moons. He floats between them in space and gets a workout by pushing and pulling on them.

DRAW this situation.

When they draw together due to their gravitational attraction, he pushes them apart. Then he pulls them back together again. But he notices something strange: when the masses fly back together, their gravitational attraction does not seem to help him pull in quite as much as this attraction hinders him from pushing them apart. In other words, the energy given up by the moons as the come together is smaller than the energy that Atlas must give the moons to push them apart.

How could that be? Shouldn't it be symmetrical? In Newtonian terms, the gravitational potential of the moons converts to kinetic energy on the way in, the kinetic energy that Atlas gives the moons on the way out is converted back to potential, and these should be equal.

But think about it from the moon's perspective. As it falls in, moon A follows the spacetime curvature, or feels the gravitational force, exerted by B. But not by B where it is at that same moment. Rather A moves as if B were at an earlier position, when B was further out. That's because information that B has moved takes time to get to A. So, if A and B move at a significant fraction of the speed of light, the attractive force of B on A will be much weaker. Weaker force implies weaker potential means less kinetic energy.

Conversely, on their way out from Atlas, A moves as if B were closer than it is now, so the force is stronger. Stonger force means stronger potential means Atlas must put in more kinetic energy.

So, moons A and B come back to Atlas with LESS energy than we sent them out with. Where did this energy (the work that Atlas did) go? Gravity waves! This is gravitational radiation, which carries energy at the speed of light. These are distortions of space that are transverse to the direction of travel.

[Think of two bodies in elliptical orbit around their center of mass.]

This effect has actually been seen, in the orbits of two neutron stars. Two neutron stars were found in orbit around one another. They are so heavy and move so quickly that the gravitational waves from these stars carries off enough energy that we can see their orbit slow down. [figure 9-7, p. 290]

Joe Taylor and Russel Hulse won the Nobel Prize for this verification of GR.

10 LIGO

It was hoped that we could directly detect gravity waves. DRAW shape of distortions. [see Cosmological Physics, fig 2-1, p. 43]

A gravity wave observatory was built to do this, called LIGO (not Lego). Laser Interferometric Gravity-wave Observatory. In 2016, it observed gravity waves!

[figure 9-8, p. 291]

Shine a laser along two orthogonal paths and recombine the beam. If a gravity wave passes by, it makes one path shorter than the other and the waves and crests of the laser light will be misaligned. We have to be extremely careful, because vibrations of trucks, much less earthquakes, can cause variations larger than predicted for gravity waves.

We quantify the effect of gravitational waves by the strain $h = \delta x/x$, which is the fractional amount that space is contracted/expanded as the wave passes through. To detect a supernova forming a neutron star in a nearby galaxy, the strain is only $h \sim 10^{-20}$. This is about the ratio of the size of an atom to the distance from Earth to Sun!