INTRODUCTION TO RELATIVITY
Winter 2018-2019
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Homework Assignment 4 Solutions

Spacetime ch.L, problems 6 (a only), 7, 8

6. Transformation of Angles

(a) In the rocket frame, the meter stick lies at angle ϕ' with respect to the x' axis. In the rocket frame $\Delta x' = (1\,\mathrm{m})\cos\phi'$ and $\Delta y' = (1\,\mathrm{m})\sin\phi'$. In the laboratory frame, the stick appears contracted along the direction of travel, thus $\Delta x = \Delta x'/\gamma = (1\,\mathrm{m})\cos\phi'/\gamma$. There is no change tranverse to the direction of travel, $\Delta y = \Delta y' = (1\,\mathrm{m})\sin\phi'$. The angle with the lab frame x axis is larger than in the rocket frame, $\tan\phi = \Delta y/\Delta x = \gamma\Delta y'/\Delta x'$, so $\phi = \tan^{-1}(\gamma\tan\phi')$. The length of the meter stick in the lab frame d is given by $d^2 = (\Delta x)^2 + (\Delta y)^2 = [(1\,\mathrm{m})\cos\phi'/\gamma]^2 + [(1\,\mathrm{m})\sin\phi']^2$. A little algebra yields $d = (1\,\mathrm{m})\sqrt{1 - v^2\cos^2\phi'}$.

7. Transformation of y-velocity

In the rocket frame, the particle moves at $v'_y = \Delta y'/\Delta t'$. Assume that the rocket moves at v_{rel} in the x direction relative to the lab frame. Apply the Lorentz transformation: No effect on distances in the transverse dimension, so in the lab frame $\Delta y = \Delta y'$. But we do have to transform the time, $\Delta t = \gamma\Delta t'$. Thus, the y-velocity in the lab frame is $v_y = \Delta y/\Delta t = \Delta y'/(\gamma\Delta t') = \sqrt{1 - v_{rel}^2}\,v'_y$. In the rocket frame, the particle has $v'_x = 0$. But in the lab frame, it moves $\Delta x = v_{rel}\Delta t$, thus $v_x = v_{rel}$.

8. Transformation of velocity direction

As suggested in the problem, solve this by transforming the space and time intervals. In the rocket frame, in time $\Delta t'$ the particle moves in the y' direction an amount

$$\Delta y' = v'\sin\phi'\Delta t'$$

and moves in the x' direction an amount

$$\Delta x' = v'\cos\phi'\Delta t'$$

Now transform $\Delta y'$ and $\Delta x'$ into the lab frame: There is no change in the transverse displacement,

$$\Delta y = \Delta y'$$

but the displacement along the direction of travel and the time does change:

$$\Delta x = \gamma\Delta x' + \gamma v_{rel}\Delta t'$$

The tangent of the angle in the lab frame is then

$$\tan\phi = \frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\gamma\Delta x' + \gamma v_{rel}\Delta t'}$$

Plug in the expressions for $\Delta y'$ and $\Delta x'$ in the rocket frame from above,

$$\tan\phi = \frac{v'\sin\phi'\Delta t'}{\gamma(v'\cos\phi'\Delta t') + \gamma v_{rel}\Delta t'}$$

The $\Delta t'$ cancels in numerator and denominator and you can also factor out v' to yield

$$\phi = \tan^{-1}\left(\frac{\sin\phi'}{\gamma(\cos\phi' + v_{rel}/v')}\right)$$

This problem differs from problem 6 because this involves the motion of a particle, rather than the appearance of an object. In the limit of very large v_{rel}, the argument of $\tan^{-1}$ goes to zero, thus $\phi \to 0$. This makes sense because the x displacement becomes huge compared to the y displacement in fixed time. In problem 6, as $v_{rel} \to \infty$, the argument of $\tan^{-1}$ becomes large, thus $\phi \to 90^\circ$