Mathematical Review

As many of you haven't taken math in a while, I thought it might be a good idea to review some of the math tools that we'll be using in this course. You should feel free to use this sheet when working on your homeworks. However, it will not be available on the midterm or the final.

ALGEBRAIC NOTATION

Algebraic expressions are just sentences, written in very compact ways. It will help your understanding considerably if you remember that, and, before trying to solve them, read them as sentences.

The Equal Sign

The simplest algebraic equations are just arithmetic:

$$x = 10 \ ,$$

for example. This is really a sentence, and it reads: "The variable, x, is equal to 10." What the equal sign means is that everything on the left side of the equation is exactly equal to everything on the right. When you have (as we do here) a *variable* on the left, it tells you what the value of that variable is. Normally, Greek or Roman letters (our alphabet is the Roman alphabet) are used for variables, but you could just as easily use any symbol you want.

Approximately Equal To

You will often see an expression like:

$$y \simeq 10$$

This reads, "The variable, y, is *approximately* equal to 10." This basically means that y may be equal to 9.5 or 11 or 8.7, or something like that. It need not be exactly equal to 10, but should be a value something like it.

> **Example:**
> I might look at an undergraduate student here at Drexel, and, for the most part say,
>
> $$a \simeq 20\text{yrs.}$$
>
> The variable, a, stands for the age of the student, and all the sentence means is that the student is approximately 20 years old. For many students, this will be a good guess.
> Notice that in this example, I didn't just say $a \simeq 20$. I also used *units* – years. In other words, I needed to specify not only how old the student was, but whether that number was how many days, months, seconds, or in this case, years old that that person was. A person who is 20 months old, for example, would hardly be a college student!

Similar To/Of Order

Sometimes, when we are being even less precise, we will use an expression like:

$$z \sim 10 \text{ feet}$$

This reads, "The variable, z, is *of order* 10 feet." Scientists, and astronomers in particular, frequently do calculations in which the number might be very, very small (0.0001, say), or it might be very, very large (a billion, for example). Sometimes, all we want to know is about how big something is.

It is for this reason that scientists use *order of magnitude* estimates. Basically, an order of magnitude is a factor of 10. If something is "good to an order of magnitude" it means that it is within $1/10th$ to $10\times$ the correct value. Thus, to say that a student in the class if ~ 10 feet tall, is a correct statement, since they will probably be 5-6 feet, rather than, on a planet of giants on which they might be 1000 feet tall.

This is obviously not the term we'd want to use in precise calculations.

Proportional To

Finally, we may know that two things are related, and, for example, we may know that when one number (the age, say, of a small child) gets larger, then another number, (the weight of that child) gets larger as well. This would be written algebraically as:

$$w \propto a$$

It reads, "The variable, w, is proportional to the variable, a." What it means that if you found two children, and the first was double the age of the second, then the first would also be double the weight of the second.

> **Example:**
> Arnold is 2 years old, and weighs 40 pounds. Betty is 3 years old. How much does she weigh?
> We'll do this in more detail later on in the course, but one way of solving expressions like this are to note that Betty is 3/2 as old as Arnold. Thus, she must weight 3/2 as much as him. $3/2 \times 40$ pounds $= 60$ pounds, so Betty would weigh 60 pounds.

Statements of proportionality aren't always so simple. For example,

$$y \propto 1/x$$

and

$$z \propto x^2$$

are also valid expressions.

The last expression, $z \propto x^2$, means that if you double x, then z goes up by a factor of 4. If you triple x, z goes up by a factor of 9, and so on.

> **Example:**
> The equation for the area of a circle is:
>
> $$A = \pi R^2$$
>
> Is there a way of writing this as a proportionality?
> Proportions are expressions which get rid of all of the constants, so a similar expression would be:
>
> $$A \propto R^2$$

We can check this. If you plug in a radius of 1 into the first expression, you find that the circle has an area of 3.14. If you plug in 2 into the first expression, you find that the circle has an area of 12.56. Note that this is $4\times$ the area of a circle with $R = 1$. In other words, by doubling the radius, you quadruple the area, which is exactly what $A \propto R^2$ means.

Exponents

An algebraic sentence will tend to have other terms in it. For example:

$$x^n$$

This reads, "x to the n-th power." And it means that you multiply x by itself n times. The variable, x, is called the *coefficient*, while n is known as the exponent.

For example, $4^3 = 4 \times 4 \times 4 = 64$.

Square Roots

The terms, x^2 and x^3, have special names. The first is read as "x-squared", while the second is "x-cubed." Scientists often square things, and, as you will see, sometimes we need to find a number, x, such that $x^2 = y$. This expression may also be written as:

$$y = \sqrt{x}$$

which reads, "The variable, y, equals the *square root* of the variable, x."

COMMONLY USED GREEK LETTERS

As I mentioned above, Greek letters are frequently used as variables in algebraic equations. Some of the most commonly used Greek letters are:

α , A - Alpha (The first letter of the Greek alphabet)

β , B - Beta (The second letter)

γ, Γ - Gamma (The third letter)

δ, Δ - Delta (The fourth letter)

θ, Θ - Theta (this will frequently be used in angles)

λ, Λ - Lambda

π, Π - Pi

ω, Ω - Omega (The final letter)

SOLVING ALGEBRAIC EQUATIONS

So, you now know how to read algebraic equations. For example, Kepler's third law (which will learn about in class) is:

$$P^2 = a^3$$

You would read this as, " The variable, P, squared is equal to the variable, a, cubed." Don't worry about what a and P represent for now. If I gave you a value of a, would you be able to tell me what value of P makes this equation true? If you can, you know how to *solve* the algebraic equation.

The trick to solving the equation is to get the equation into a form like:

$$P = ??$$

The rule is that whatever you do to the left side of the equation you need to do the same thing to the right side. So, if you take the original equation, $P^2 = a^3$, how do you get just, P on the left? By taking the square root!

Left Side:

$$\sqrt{P^2} = P$$

Right Side:

$$\sqrt{a^3}$$

So,

$$P = \sqrt{a^3}$$

(Some of you may recall that $\sqrt{a^3}$ can also be written as $a^{3/2}$, but don't worry about that now.)

So, if $a = 4$, for example, what is P?

$4^3 = 64$ (as we saw before). Plugging the $\sqrt{64}$ into our calculators, we get: $P = 8$.

Let's do a bunch of examples.

Example 1:
Solve for y.

$$x^2 + y = 4$$

Subtract x^2 from both sides.
Left Side:

$$x^2 + y - x^2 = y$$

Right Side:

$$4 - x^2$$

So:

$$y = 4 - x^2$$

Example 2:
Solve for y.

$$4y = 28$$

Divide by 4.
Left Side:

$$4y/4 = y$$

Right Side:

$$28/4 = 7$$

So:

$$y = 7$$

Example 3:
Solve for y.

$$-4y + x = 3$$

Subtract x.
Left Side:

$$-4y + x - x = -4y$$

Right Side:

$$3 - x$$

Divide by -4.
Left Side:

$$-4y/(-4) = y$$

Right Side

$$(3 - x)/(-4) = -3/4 + x/4$$

Notice that dividing a negative by a negative gives a positive, and dividing a negative by a positive gives a negative.
So,

$$y = x/4 - 3/4$$

Example 4:
Solve for y.

$$y = \frac{1}{\frac{1}{x}}$$

This is a trick! $1/(1/x) = x$. So $y = x$. Remember, if you have a fraction within a fraction, then the "downstairs" (denominator) of the "downstairs" (in this case, x) comes up to the "upstairs" (numerator).

UNITS

In general, we will want to use units in this class. They are *extremely* important, as I tried to illustrate above. In general, there are three types of units which will come up again and again:

Length: meters (m), centimeters (cm), feet (ft.), light-years (ly), parsecs (pc), astronomical units (AU), inches (in), etc.

At this point, I only expect you to know the metric system:

$2.54cm = 1in.$

$1000mm = 1m$

$100cm = 1m$

$1000m = 1km$

Times: seconds (s), years (yr), Gigayears (Gyr), etc.

The relations are:

60s=1 min.

60 min.=1 hr.

24 hr=1 day

365 days=1 yr

31,600,000 s $\simeq$ 1 yr.

Mass: kilograms (kg), grams (g), Solar-Masses ($M_\odot$), etc.

1000 g=1 kg

Some of the units above will seem unfamiliar. I will explain over the course, for example, what a parsec is. However, in any event, you need to know what to do when you find some units in an algebraic expression.

For example, imagine you have an equation like $x = 27cm$. However, imagine that I asked for the answer in m. What can you do?

- You may always multiply one side of an equation by a fraction, so long as the numerator and denominator of the fraction are equal.

What I mean is this: $100cm = 1m$. So, the fraction:

$$\frac{1m}{100cm}$$

is just like multiplying by 1. As you recall, if you multiply by 1, nothing changes so:

$$x = \frac{27cm}{1}\frac{1m}{100cm}$$

leaves the equation just as true as before.

- If you have the same units in the numerator and denominator, they can cancel out.

So

$$x = \frac{27\cancel{cm}}{1}\frac{1m}{100\cancel{cm}} = \frac{27m}{100} = 0.27m$$

Example 1:

$$y = \frac{10m^3}{cm^2}$$

Let's solve in units of m. Now, notice that there are 2 factors of cm on the bottom, so:

$$y = \frac{10m^{\cancel{3}\,1}}{\cancel{cm^2}} \times \frac{100\cancel{cm}}{\cancel{m}} \times \frac{100\cancel{cm}}{\cancel{m}} = 10 \times 100 \times 100m = 100,000m$$

Note that we had to multiply by the conversion factor twice. Also, notice that there are two m terms canceled on both the top and bottom, and two cm terms canceled on the top and bottom.

Example 2:
Consider

$$y = \frac{x}{z^2}$$

If $x = 10s$ and $z = 10m/s$, how do we solve this?
Well, first, note that:

$$z^2 = (10m/s)^2 = 10^2\frac{m^2}{s^2} = 100\frac{m^2}{s^2}$$

Remember, if you square something with units, you have to square all of the units.
So,

$$y = \frac{10s}{100\frac{m^2}{s^2}}$$

However, remember that things when you have a fraction within a fraction, that things in the "downstairs" of the "downstairs" go "upstairs". This is the case with the s^2.

$$y = \frac{10s^3}{100m^2} = 0.1\frac{s^3}{m^2}$$

Example 3:
Consider the equation

$$y = \frac{200(28m^2)(10cm)(10g)}{20cm^2}$$

How do we solve for this, including units?
Step 1: Group all of the coefficients and units together

$$y = \frac{200 \times 28 \times 10 \times 10}{20} \frac{m^2 \cdot cm \cdot g}{cm^2}$$

Step 2: Multiply all of the coefficients

$$y = 28,000 \frac{m^2 \cdot cm \cdot g}{cm^2}$$

Step 3: Simplify the units.
Notice that on the top has three units of length, while the bottom has only two. One of the centimeters from the bottom can be used to cancel with a centimeter from the top.

$$\begin{aligned} y &= 28,000 \frac{m^2 \cdot \cancel{cm} \cdot g}{cm^{\cancel{2}\ 1}} \\ &= 28,000 \frac{m^2 \cdot g}{cm} \\ &= 28,000 \frac{m^{\cancel{2}\ 1} \cdot g}{\cancel{cm}} \frac{100\cancel{cm}}{\cancel{m}} \\ &= 28,000 \times 100\ g \cdot m \\ &= 2,800,000\ g \cdot m \end{aligned}$$

SCIENTIFIC NOTATION

Notice that in the above example, the numbers are starting to get pretty big. That happens a lot in astronomy, since we deal with very big (and occasionally) very small, numbers. It is because of this that we will usually deal with *scientific notation*.

Basic Rules

Let's begin with some examples:

5.8×10^3

-3.9×10^{33}

6.52×10^{-7}

What do all of these numbers have in common? They all have an ordinary number (between 1 and 10) times a 10 with an exponent. The nice thing about 10s are that if you multiply 10^n power, you get a 1 with n zeroes after it. For example,

$10^3 = 1000$

If you are confused by this, remember the definition of exponents. Remember, $10^3 = 10 \times 10 \times 10$. Plug those numbers into your calculator and verify that you get 1000.

Likewise, if the exponent is negative, the exponent tells you how many digits you have to the *right* of the decimal point:

$10^{-2} = 0.01$

Negative exponents are a bit confusing. Remember from high school that $10^{-n} = 1/10^n$. So, $10^{-2} = 1/10^2$. $10^2 = 100$. Thus, $10^{-2} = 1/100 = 0.01$. Get it?

There is one more case. What about 10^0? Normally, we don't even bother writing this, because

$10^0 = 1$

(It's a 1 with zero zeroes.)

You can express any number in scientific notation.

For example, 59,000:

Step 1:

Count how many places the decimal point would have to move so that only one number would be to the left of it. In this case, it would have to move over 4 places. This will be your exponent.

Step 2:

After moving the decimal point, look at the number you have left, this is your coefficient.

$59000 \rightarrow 5.9$

Step 3:

Combine the terms.

$$59,000 = 5.9 \times 10^4$$

You can verify this with your calculator. $10^4 = 10000$, and $5.9 \times 10,000 = 59,000$.

Example 1:
What is 45 in scientific notation?
Moving the decimal point 1 place to the left leaves you with 4.5, the exponent is 1 and the coefficient is 4.5.

$$45 = 4.5 \times 10^1$$

Example 2:
What is 0.0089 in scientific notation?
In this case, you have to move the decimal place 3 places, *to the right*, so the exponent is -3. The coefficient is 8.9, so

$$0.0089 = 8.9 \times 10^{-3}$$

Multiplication and Division

Another nice property of powers of 10 is that:

$$10^n \times 10^m = 10^{n+m}$$

Verify, for example, that $10^2 \times 10^3 = 10^5$.

The fact that you can add exponents makes multiplying big numbers simple.

Example:

$$\begin{aligned}(5.9 \times 10^{10}) \times (3 \times 10^{20}) &= 5.9 \times 3 \times 10^{10+20} \\ &= 17.7 \times 10^{30} \\ &= 1.77 \times 10^{31}\end{aligned}$$

Notice that last step. Since 17.7 is bigger than 10, I had to move the decimal place one more time, and I added 1 more to my exponent.

Division also works nicely:

$$\frac{10^n}{10^m} = 10^{n-m}$$

Example:

$$\begin{aligned}\frac{6 \times 10^{11}}{2 \times 10^5} &= 6/2 \times 10^{11-5} \\ &= 3 \times 10^6\end{aligned}$$

Powers

Finally, squaring a power of 10, for example, is very simple:

$$(10^n)^2 = 10^{2n}$$

So, for example, $(10^3)^2 = 10^6$. Verify that $1000^2 = 1,000,000$.

Special Names

As a final topic, you should be aware that some powers of 10 have special names. For examples, if we are talking about units of length:

$10^{-9}m = 1$ nanometer (nm)

$10^{-6}m = 1$ micrometer (μm)

$10^{-3}m = 1$ millimeter (mm)

$10^{-2}m = 1$ centimeter (cm)

$10^{0}m = 1$ meter (m)

$10^{3}m = 1$ kilometer (km)

$10^{6}m = 1$ Megameter (Mm)

$10^{9}m = 1$ Gigameter (Gm)

These prefixes work with units of mass, time, or any other unit. For example, a very common unit of time is the Gigayear, Gy, which is 10^9 years.