Physics 432/750: Cosmology
Winter 2003-2004
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Problem Set 2 Solutions

1. Field of View of the Hubble Space Telescope

The Hubble Space Telescope images an area on the sky that is approximately 2.5 minutes of arc across. Leave your answers to this problem in terms of the Hubble constant parameter h. [Recall units of angle: 60 arcseconds/arcminute, 60 arcminutes/degrees, 180 degrees/π radian.]

(a) What is the proper size in Mpc of the HST field of view at redshifts $0.5, 1, 2, 4$ for $\Omega_{matter} = 1, \Omega_{vac} = 0$ and for $\Omega_{matter} = 0.3, \Omega_{vac} = 0.7$? You may, if you choose, answer this by presenting plots for both cosmologies.

The proper size of an object that subtends an angle θ at redshift z is

$$l = \theta D_A(z)$$

where $D_A(z)$ is the angular diameter distance, related to the comoving coordinate distance $D(z)$ by $D_A(z) = D(z)/(1+z)$. Obviously, θ must be dimensionless, i.e. in radians. The $\Omega_{matter} = 1$ case is precisely described by Mattig's formula, which yields

$$l = \frac{\theta D(z)}{1+z} = \frac{\theta 2c}{H_0} \frac{z + [1 - \sqrt{1+z}]}{(1+z)^2} = 6000\theta h^{-1}\mathrm{Mpc}\frac{z + [1 - \sqrt{1+z}]}{(1+z)^2}$$

For $\theta = 2.5' = 7.3 \times 10^{-4}\mathrm{rad}$, the proper size of the HST field of view is 0.536, 0.641, 0.617, and 0.458h^{-1} Mpc for $z = 0.5, 1, 2,$ and 4, respectively. Thus, the field of view is of order 1 Mpc proper size over a large range of redshift – the angular size-distance relation does not change much over this range. For the $\Lambda \neq 0$ case, Pen's formula works nicely (it could be used for both cases – Pen's approximation matches the exact form of Mattig's relation quite well), but note that is formula is for the *luminosity* distance, not the angular diameter distance (recall they are related by $D_A(z) = D_L(z)/(1+z)^2$). We obtain proper sizes of 0.645, 0.857, 0.894, and 0.743h^{-1} Mpc for $z = 0.5, 1, 2,$ and 4, respectively.

If for some reason you left your answers in term of arcminutes, rather than convert angles to radians, then your answers should larger than the above by 3438.

(b) What is the comoving size of the field at $z = 1$ for both cosmologies?

The comoving size is larger by the factor $(1+z)$, to account for the expansion, thus 1.28 and 1.71h^{-1} Mpc, for $\Omega_{matter} = 1$ and $\Omega_{vac} = 0.7$, respectively.

2. Angular Sizes of Galaxies

Using HST, the Hubble Deep Field observation (two weeks of pointing at the same place on the sky!) reveals galaxies that range in redshift from $z = 0.2$ to $z = 4$ and in angular size from 0.1 to 5 arcseconds. Leave your answers to this problem in terms of the Hubble constant parameter h.

(a) For each of the two cosmologies in problem 1, make plots of $\theta(z)$ (the angular size on the sky in arcseconds) for several choices of l_{gal} (the proper physical size) in the range $l = 0.5$

kpc to $l = 10$ kpc (recall that one parsec is 3 light years). Discuss the relationship between angular size and proper size of the galaxies. How is this complicated by cosmology?

For fixed projected size on the sky θ and proper object size l and fixed cosmology, the angular diameter-redshift relation has two possible solutions for the redshift. Varying the cosmology shifts these solutions to larger or smaller redshift. For very low Ω_{matter} universes, particularly if Ω_{vac} is small or zero, the second (high-z) solution shifts to well beyond where galaxies could yet have formed (much less be detected at optical wavelengths even if they exist), so there is only one likely solution. If the intrinsic proper length is not known, then the angular size alone allows a continuum of possible solutions of different proper size and redshift, even for fixed cosmology.

(b) What measurements or other information about these galaxies would allow us to disentangle this ambiguity?

Obviously, a spectroscopic redshift of a galaxy yields a unique proper size for a particular cosmology. Lacking a direct z measurement, the colors of a galaxy are helpful, since the redshift makes distant galaxies redder. Evolution complicates this, since star formation must begin at some high redshift, thus distant galaxies might have substantially higher rates of star formation, making them intrinsically bluer than galaxies at $z = 0$. The surface brightness of the galaxy might also be a clue; the strong $(1+z)^{-4}$ bolometric surface brightness dependence implies that, in the absence of evolution of galaxies, galaxies at large redshift will appear to have lower surface brightness. But the higher star formation rates may give distant galaxies higher intrinsic surface brightness. And there is a strong bias against detecting low surface-brightness galaxies above the foreground surface brightness of the night sky. Thus, studying the high-redshift universe is, to say the least, non-trivial.

3*. Blackbody Radiation

An object emits blackbody radiation of temperature T in its own rest frame. We see it at redshift z and subtending a solid angle Ω (here Ω is solid angle, not the density parameter!).

(a) What flux do we measure?

Large hints for this problem are in chapter 9, p.290 of the text. The total observed flux is

$$S = \Omega \int I_\nu^{obs} d\nu_{obs}$$

For an object at redshift z, the rest-frame frequency is $\nu = \nu_{obs}(1+z)$. The phase-space density of photons is a constant, proportional to I_ν/ν^3. Thus, we can rewrite the observed flux

$$S = \Omega \int \left(\frac{I_\nu^{obs}}{\nu_{obs}^3}\right) \nu_{obs}^3 d\nu_{obs} = \Omega \int \left(\frac{I_{nu}}{\nu^3}\right) \frac{\nu^3 d\nu}{(1+z)^4}$$

This simplifies to

$$S = \frac{\Omega}{(1+z)^4} \frac{\sigma}{\pi} T^4$$

(b) What if the redshift is due to Doppler motion of a local object instead of cosmological?

Of course, conservation of phase-space density applies regardless of the source of the redshift, thus a redshift z caused by a Doppler shift yields the same formula for the flux S.

4. Uncertainty in the distance scale

Suppose that a series of four different standard candles are used to step out along the cosmic distance ladder from our Galaxy to distances far enough to accurately measure the true expansion rate. Each standard candle has an uncertainty of $\Delta m = 0.2$ magnitudes in its intrinsic magnitude. Show that, by varying the calibration of each of the standard candles, it is possible for the measured Hubble constant to differ from its nominal value by a factor of ~ 0.7 to ~ 1.4. (Recall the definition of astronomical magnitudes, $m = -2.5 \log f + \text{constant}$ where f is the received flux.)

Since $f \propto Lr^{-2}$, $m = -2.5 \log L + 5 \log r + \text{constant}$, so an uncertainty Δm corresponds to an uncertainty in the distance

$$r \propto 10^{\Delta m/5}$$

Thus an uncertainty $\Delta m = \pm 0.2$ in the absolute magnitude of an object translates to a multiplicative uncertainty in its distance of $1.096r$ or $0.912r$. If the distance ladder is constructed by successive calibrations of the distance from one step to the next, then the uncertainty in the final distance (to the outermost object) is the product of the errors in all four steps. For all $\Delta m = 0.2$, this is a factor $(1.096)^4 = 1.44$. For all $\Delta m = -0.2$, we underestime the distance by a factor 0.692. The Hubble constant is estimated by taking the ratio of the recession velocity to the estimated distance, $H = v/r$, thus the error in H is inversely proportional to the error in r.

5. Limits on the parallax method

From Earth, optical telescopes can resolve differences in angle of at best 0.5 arcseconds. With this resolution limit, to what distance can we use trigonometric parallax to measure distances? What if we could measure differences in angle of 0.01 arcseconds? (Recall that there are 206,265 arcseconds in a radian.)

An angle $\theta = 0.5$ arcsec corresponds to 2.4×10^{-6} rad. Using the Earth's orbit as the baseline for the triangle, 1 A.U. $= 1.4 \times 10^{11}$ m, the maximum distance to which parallax would be accurate is

$$D = \frac{1.4 \times 10^{11} \text{m}}{2.4 \times 10^{-6}} = 6.2 \times 10^{16} \text{m}$$

or 2 parsecs. Recall that the *definition* of a parsec is the distance at which the parallax is one arcsecond. For higher resolution, $\theta = 0.01$ arcsecond, the maximum distance is simply 50 times larger, or 100 parsecs.

6. Peculiar velocities and the Hubble constant

The observed recession velocity of galaxies is the sum of their Hubble flow velocity, $v = Hr$, and the line-of-sight component of their "peculiar" velocity, due to local gravitational effects. Suppose that we estimate H_0 by measuring both the distance and redshift for a single object. How does the error in the Hubble constant depend on the r.m.s. peculiar velocity of this type of object? At what distance can we get 5% accuracy in H_0 if $v_{pec} = 200$ km s^{-1}?

Our estimate of the Hubble constant is $H_{est} = v_{obs}/r_{obs}$. The true Hubble constant is $H_0 = v_{cosm}/r$ For relatively small distances, the cosmological recession velocity is $v_{cosm} = H_0 r$. The observed velocity includes the radial component of the object's peculiar velocity

$v_{obs} = v_{cosm} + v_r$. Thus the error in the estimate is

$$\delta H_{est} = \frac{v_r}{r}$$

thus the uncertainty in the Hubble constant is proportional to the peculiar velocity divided by the distance to the object. We can get more accurate estimates by looking at objects at larger distance. To get 5% precision, we require

$$\frac{(v_{pec}/r)}{H_0} < 0.05$$

thus

$$r > \frac{v_{pec}}{0.05 H_0}$$

For $v_{pec} = 200$ km s^{-1} and $H_0 = 100h$ km s^{-1} Mpc^{-1}, this requires $r > 40h^{-1}$ Mpc.

7. Age of the universe and constraints on cosmological parameters

Make a 2D figure that shows the dependence of the age of the universe on Ω_{matter} and Ω_{vac}. Indicate the bounds on this 2D space that are set by age estimates of the Solar system, white dwarfs, and globular clusters. How does the region of $\Omega_{matter}, \Omega_{vac}$ that is allowed by these constraints compare with the region favored by other observations? (See your notes from class, figure 3.5 in the text, and the "Cosmic Triangle" article.)

The equation

$$t_0 = \frac{2}{3} t_H (0.7\Omega_M + 0.3 - 0.3\Omega_{vac})^{-0.3}$$

relates the age of the universe to the Hubble time ($t_H = 1/H$) and the cosmological parameters Ω_M and Ω_{vac}. For fixed t_0 this yields the equation of a straight line in the $\Omega_M - \Omega_{vac}$ plane,

$$\Omega_{vac} = \frac{7}{3}\Omega_M + 1 - \frac{10}{3}\left(\frac{2t_H}{3t_0}\right)^{10/3}$$

The bounds on the age of the universe from local observations are roughly $t > 9$ Gy from WD's, $t > 10$ Gy from the Solar system, and $t > 12$ Gy from globular clusters. The corresponding line for each of these forms a lower bound in the Ω_M, Ω_{vac} plane. Allowed models lie *above* these lines – the universe must be at least as old as its contents. An Einstein-de Sitter model is too young for both Solar system and globular cluster constraints. A pure open universe with small enough Ω_M can just barely squeeze above all these bounds. $\Omega_M = 0.3, \Omega_{vac} = 0.7$ satifies all these bounds and is consistent with most other astrophysical constraints.